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New Canonical Forms for Enumerating Fuzzy/C Switching Functions

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Abstract

Logic functions such as fuzzy switching functions and multiple-valued Kleenean functions, that are models of Kleene algebra have been studied as foundation of fuzzy logic. This paper deals with a new kind of functions - fuzzy switching functions with constants - which have features of both the above two kinds of functions In this paper, we propose new canonical forms for enumerating them. They are much useful to estimate simply the number of fuzzy switching functiuons with constants.

Keywords: fuzzy switching function with constants, canonical form, enumeration

1. Introduction

Fuzzy switching functions[1,2,9,12] have been studied since the beginning of study on fuzzy logic, which are defined as infinite-valued logic functions and represented in logic formulas. In the definition of logic formulas which represent fuzzy switching functions, constants take an only value of either zero(0) or one(1) although any variable can take any value in the closed interval [0,1]. From algebraic point of view, fuzzy switching functions are a model of Kleene algebra. Kleene algebra is also called fuzzy algebra.

On the other hand, in the field of multiple-valued logic, multiple-valued Kleenean functions[13-16] have been studied by many researchers. Multiple-valued Kleenean functions are also a model of Kleene (fuzzy) algebra, and are represented in logic formulas. In the case of m-valued, both constants and variables of multiple-valued Kleenean functions take discrete values such as $0, 1/(m-1), \cdots, (m-2)/(m-1), 1$. Therefore, multiple-valued Kleenean functions are not infinite-valued logic functions.

As mentioned in the above, both fuzzy switching functions and multiple-valued Kleenean functions are a model of Kleene (fuzzy) algebra. As generalization of these logic functions, fuzzy switching functions with constants (for short, Fuzzy/C switching functions) whose constants and variables are allowed to take any arbitrary value in the closed interval [0,1] have been studied by the authors[3-6,10]. Among these works, enumeration problem is one of the remained works. This paper describes on the problem.

At first, it is clarified that the radix is essentially finite although Fuzzy/C switching functions are infinite-valued logic functions. That is, the number of truth values and constants is essentially finite. And, it is less than or equal to 3^n in the case of n-variable (Theorem 6).

Secondly, new canonical forms for enumerating Fuzzy/C switching functions are proposed (Theorem 7). These canonical forms do not contain complementary phrases and complementary clauses. Therefore, it is shown that the enumerating problem of Fuzzy/C switching functions can result in that of binary switching functions (Theorem 10).

2. Fuzzy switching functions with constants

2.1 Fundamental definitions

[Definition 1] *Logic formulas* are defined as follows: (1) Constants $c_1, c_2, \cdots, c_\ell$ and variables $x_1, x_2, \cdots, x_n$ are logic formulas. (2) If F and G are logic formulas, $\sim F$, $F \cdot G$ and $F \vee G$ are logic formulas, respectively. (3) Logic formulas are only defined by (1) and (2). //

[Definition 2] Let $V = [0,1]$. Then mapping $f: V^n \to V$ which is represented in a logic formula is called an n-variable *fuzzy switching function with constants* (for short, *Fuzzy/C function*). Where logic connectives are interpreted as $\sim x = 1 - x$, $x_1 \cdot x_2 = \min(x_1, x_2)$ and $x_1 \vee x_2 = \max(x_1, x_2)$, respectively. And constants $c_1, c_2, \cdots, c_\ell$ are interpreted as 0-variable functions. //

Hereafter, we use $V = [0,1]$, $V_3 = \{0, 0.5, 1\}$ and $V_2 = \{0,1\}$ as notation of sets of truth values, respectively. And the notation of conjunction($\cdot$) is often omitted from logic formulas.

[Example 1] For two variables, the following f and g are Fuzzy/C switching functions:

$$f = \sim x_1 x_2 \vee 0.8 \sim x_1 \vee 0.3 \sim x_1 \sim x_2 \vee x_1 x_2 \sim x_2$$

$$g = (\sim x_1 \vee x_2) \cdot (0.3 \vee \sim x_1 \vee \sim x_2) \cdot (0.8 \vee x_1 \vee x_2) \cdot (x_1 \vee \sim x_1 \vee \sim x_2) \quad //$$

[Definition 3] Let f be a Fuzzy/C switching function. And let constants which appear in the logic formula of f be $c_1, c_2, \cdots, c_t$. Then we define the following $R_0(f)$ as the *base set* of f:

$$R_0(f) = \{0, 0.5, 1\} \cup \{c_1, c_2, \cdots, c_t\} \cup \{1-c_1, 1-c_2, \cdots, 1-c_t\} \qquad //$$

It is clear that cardinality of the base set of any Fuzzy/C switching function is odd.

Logic formulas of Fuzzy/C switching functions can be represented in disjunctive normal forms and/or conjunctive normal forms as good as binary switching functions.

[Definition 4] Phrases (clauses) which appear in disjunctive (conjunctive) normal forms of Fuzzy/C switching functions are classified as follows:

(Type P1 (C1)) Phrases (Clauses) which have a constant greater than (less than) 0.5 and don't have any pair of variables such as $x_i \cdot \sim x_i$ ($x_i \vee \sim x_i$).

(Type P2 (C2)) Phrases (Clauses) which have a constants less than (greater than) or equal to 0.5 and don't have any pair of variables such as $x_i \cdot \sim x_i$ ($x_i \vee \sim x_i$).

(Type P3 (C3)) Phrases (Clauses) which have a constant less than (greater than) or equal to 0.5 and have at least one pair of variables such as $x_i \cdot \sim x_i$ ($x_i \vee \sim x_i$). If a constant is omitted, we consider the phrase (clause) to have 0.5.

A phrase (clause) which is of Type P1 (C1) and Type P2 (C2) is called a *simple phrase (clause) with a constant* (for short, a *simple phrase (clause)*). A phrase (clause) which is of Type P3 (C3) is called a *complementary phrase (clause) with a constant* (for short, a *complementary phrase (clause)*). And phrases of Type P2 and Type P3 (clauses of Type C2 and Type C3) can be expanded into a disjunction of phrases (conjunction of clauses) which have all variables because $x_i \sim x_i \leq 0.5 \leq x_j \vee \sim x_j$ always holds. A phrase of Type P2 (clause of Type C2) which have all variables is called a *minterm (maxterm) with a constant* (for short, a *minterm (maxterm)*). A phrase of Type P3 (clause of Type C3) which have all variables is called a *complementary minterm (maxterm) with a constant* (for short, *complementary minterm (maxterm)*). //

[Example 2] In the case of two variables,

(1) $0.8x_1$ and $\sim x_1x_2$ are simple phrases of Type P1, respectively.

(2) $0.2x_1$ is a simple phrase of Type P2. This phrase can be expanded into a disjunction of two minterms as follows:

$$0.2x_1 = 0.2x_1(x_2 \vee \sim x_2) = 0.2x_1x_2 \vee 0.2x_1 \sim x_2.$$

(3) $x_1 \sim x_1$ is a complementary phrase of Type P3. This phrase can be expanded into a disjunction of two complementary minterms as follows:

$$x_1 \sim x_1 = x_1 \sim x_1(x_2 \vee \sim x_2) = x_1 \sim x_1x_2 \vee x_1 \sim x_1 \sim x_2. \qquad //$$

[Definition 5] If a logic formula of a Fuzzy/C switching function f is represented in a disjunction of simple phrases of Type P1, minterms of Type P2 and complementary minterms of Type P3 (a conjunction of simple clauses of Type C1, maxterms of Type C2 and complementary maxterms of Type C3), then f is said to be represented in a *canonical disjunctive (conjunctive) normal form.* //

[Example 3] The Fuzzy/C switching function f in example 2 is represented in the canonical disjunctive normal form of f. And g is the canonical conjunctive normal form of f. //

[Definition 6] (A *partially ordered relation with respect to ambiguity*) If $a, b \in V$, then $a \prec b$ is equivalent to $0 \leq a \leq b \leq 0.5$ or $0.5 \leq b \leq a \leq 1$. And this partially ordered relation $\prec$ can be extended into n-dimensional case. Let $A = (a_1, \cdots, a_n) \in V^n$ and $B = (b_1, \cdots, b_n) \in V^n$, respectively. Then $A \prec B$ is equivalent to $\forall i(a_i \prec b_i)$. //

2.2 Fundamental properties

[Theorem 1] [6] (*A monotonous property with respect to ambiguity*) Let f be a Fuzzy/C switching function. Then if $A \prec B$, $f(A) \prec f(B)$ holds. //

[Theorem 2] [6] Let $R_0(f)$ be the base set of a Fuzzy/C switching function f. From algebraic point of view, f is equivalent to a multiple-valued Kleenean function of $|R_0(f)|$-valued. //

[Example 4] Let f be the given 3-variable Fuzzy/C switching function as follows:

$$f = x_1 \sim x_2 \vee 0.8x_2x_3 \vee 0.4x_3 \sim x_3$$

Then $R_0(f) = \{0, 0.2, 0.4, 0.5, 0.6, 0.8, 1\}$. Because $|R_0(f)| = 7$, the following 7-valued multiple-valued Kleenean function g whose truth value is $V_7 = \{0, 1/6, 1/3, 1/2, 2/3, 5/6, 1\}$ is corresponding to f:

$$g = x_1 \sim x_2 \vee (5/6)x_2x_3 \vee (1/3)x_3 \sim x_3 \qquad //$$

[Theorem 3][6] Let f and g be n-variable Fuzzy/C switching functions, respectively. Then $\forall A \in V^n (f(A) = g(A))$ is equivalent to $\forall B \in V_3^n (f(B) = g(B))$. //

[Theorem 4][6] Any Fuzzy/C switching function can be represented in its canonical disjunctive (conjunctive) normal form uniquely without considering the order of phrases (clauses). //

According to thorem 3, it is shown that properties for functional value of any Fuzzy/C switching function can be considered as its domain is restricted to V_3^n. In the bellow, we consider on relation between subsets of V_3^n and the canonical disjunctive (conjunctive) normal form of a given Fuzzy/C switching function.

[Definition 7] Let c be a constant and α (β) be a phrase (clause).

$(a_1, a_2, \cdots, a_n) \in V_3^n$ and phrases $\alpha = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ (clauses $\beta = c \vee x_1^{a_1} \vee x_2^{a_2} \vee \cdots \vee x_n^{a_n}$) of Type P1 (C1) and Type P2 (C2) are corresponding to each other as follows:

$$x_i^{a_i} = \begin{cases} x_i & (\sim x_i) & for\ a_i = 1 \\ 1 & (0) & for\ a_i = 0.5 \\ \sim x_i & x_i & for\ a_i = 0 \end{cases} \qquad (1)$$

$(a_1, a_2, \cdots, a_n) \in V_3^n - V_2^n$ and phrases $\alpha = c \cdot x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ (clauses $\beta = c \vee x_1^{a_1} \vee x_2^{a_2} \vee \cdots \vee x_n^{a_n}$) of Type P3 (C3) are corresponding to each other as follows:

$$x_i^{a_i} = \begin{cases} x_i & (\sim x_i) & for\ a_i = 1 \\ x_i \sim x_i & (x_i \vee \sim x_i) & for\ a_i = 0.5 \\ \sim x_i & x_i & for\ a_i = 0 \end{cases} \qquad (2)$$

//

Let the base set of a Fuzzy/C switching function f be $R_0(f) = \{c_0, \cdots, c_K, c_{K+1}, 1-c_K, \cdots, 1-c_0\}$ (where $c_0 = 0$, $c_{K+1} = 0.5$, $0 \le c_i < 0.5 (i = 0, 1, \cdots, K)$ and $c_i < c_{i+1}$, respectively.). And let the canonical disjunctive normal form of f be given as follows:

$$f = (\alpha_1^{c_0} \vee \cdots \vee \alpha_a^{c_0}) \vee \cdots \vee (\alpha_1^{c_K} \vee \cdots \vee \alpha_b^{c_K})$$
$$\vee (\alpha_1^{c_{K+1}} \vee \cdots \vee \alpha_d^{c_{K+1}}) \vee (\alpha_1^{1-c_K} \vee \cdots \vee \alpha_e^{1-c_K}) \vee \cdots$$
$$\vee (\alpha_1^{1-c_0} \vee \cdots \vee \alpha_h^{1-c_0})$$

Where $\alpha_j^{1-c_i} (i = 1, \cdots, K)$ is a simple phrase which has a constans $1 - c_i$, and $\alpha_j^{c_i} (i = 1, \cdots, K+1)$ is either a minterm or a complementary minterm which has a constant c_i.

Similarly, let the canonical conjunctive normal form of f be given as follows:

$$f = (\beta_1^{c_0} \cdots \beta_r^{c_0}) \cdots \cdots (\beta_1^{c_K} \cdots \beta_s^{c_K}) \cdot (\beta_1^{c_{K+1}} \cdots \beta_t^{c_{K+1}})$$
$$\cdot (\beta_1^{1-c_K} \cdots \beta_u^{1-c_K}) \cdots (\beta_1^{1-c_0} \cdots \beta_v^{1-c_0})$$

Where $\beta_i^{1-c_i} (i = 1, \cdots, K+1)$ is a simple clause which has a constant c_i, and $\beta_i^{1-c_i} (i = 1, \cdots, K+1)$ is either a maxterm or a complementary maxterm which has a constant $1 - c_i$.

And let $f^{-1}(y)$ be a subset of V_3^n defined as follows:

$$f^{-1}(y) = \{A \in V_3^n \mid f(A) = y, y \in R_0(f)\}$$

Moreover, let T be a function which is mapping from set of phrases and clauses to elements in V_3^n according to definition 7. Then the following theorem holds.

[Theorem 5][6]

(1) $f^{-1}(1) = \bigcup_j \{A \in V_3^n \mid A \prec T(\alpha_j^{1-c_0})\}$

(2) $f^{-1}(0) = \bigcup_j \{A \in V_3^n \mid A \prec T(\beta_j^{c_0})\}$

(3) For $i = 1, 2, \cdots, K$,

$$f^{-1}(c_i) = \left(\bigcup_j \{A \in V_3^n \mid A \succ T(\alpha_j^{c_i})\}\right) \cap \left(\bigcup_j \{A \in V_3^n \mid A \prec T(\beta_j^{c_i})\}\right)$$

(4) For $i = 1, 2, \cdots, K$,

$$f^{-1}(1 - c_i) = \left(\bigcup_j \{A \in V_3^n \mid A \prec T(\alpha_j^{1-c_i})\}\right) \cap \left(\bigcup_j \{A \in V_3^n \mid A \succ T(\beta_j^{1-c_i})\}\right)$$

(5) $$f^{-1}(0.5) = \left(\bigcup_j \{A \in V_3^n \mid A \succ T(\alpha_j^{c_{K+1}})\}\right) \cup \left(\bigcup_j \{A \in V_3^n \mid A \succ T(\beta_j^{c_{K+1}})\}\right) \cup \left(\{A \in V_3^n \mid \forall B \in \bigcup_{i=0}^{K} f^{-1}(c_i), B \prec A\} \cap \{A \in V_3^n \mid \forall B \in \bigcup_{i=0}^{K} f^{-1}(1 - c_i), B \prec A\}\right)$$

(Proof is omitted.)

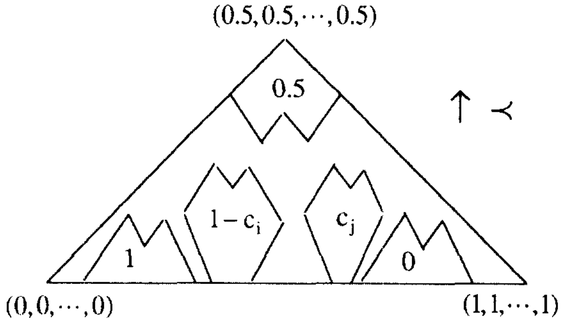

Fig.1 Partition of the domain V_3^n by value of a Fuzzy/C function.

What theorem 5 means is shown in Fig.1.

According to theorem 5, it is shown that elements of V_3^n which are corresponding to phrases in the canonical disjunctive normal form of f and clauses in the canonical conjunctive normal form of f are corresponding to either maximal elements of $f^{-1}(y)$ or minimal elements of $f^{-1}(y)$. It is clear that these subsets $f^{-1}(1)$, $f^{-1}(0)$, $f^{-1}(0.5)$, $f^{-1}(c_i)$ and $f^{-1}(1 - c_i)$ divide V_3^n as direct sum. That is,

$$f^{-1}(1) \cup f^{-1}(0) \cup f^{-1}(0.5)$$
$$\cup \left(\bigcup_i f^{-1}(c_i)\right) \cup \left(\bigcup_i f^{-1}(1 - c_i)\right) = V_3^n$$

[Theorem 6] For n-variable Fuzzy/C switching functions, the number of constants which appear in their logic formulas is finite, and is less than or equal to 3^n.

(Proof) As any constant c_i is arbitrary value in $[0, 0.5)$, the number of candidates of c_i is infinite. Therefore, the number of subsets $f^{-1}(c_i)$ and $f^{-1}(1 - c_i)$ are infinite. However, the number of elements in $f^{-1}(1) \cup f^{-1}(0) \cup f^{-1}(0.5) \cup \left(\bigcup_i f^{-1}(c_i)\right) \cup \left(\bigcup_i f^{-1}(1 - c_i)\right)$ is equal to the number of elements in V_3^n. Consequently, although there are infinite number of subsets $f^{-1}(c_i)$ and $f^{-1}(1 - c_i)$, only finite number of subsets $f^{-1}(c_i)$ and $f^{-1}(1 - c_i)$ are not empty, and the others are empty. //

According to theorem 6, if a Fuzzy/C switching function f is given, then $|R_0(f)| \le 3^n$ holds. Moreover, according to theorem 2, set of Fuzzy/C switching functions whose base set is $R_0(f)$ are equivalent to set of multiple-valued Kleenean functions whose radix are $|R_0(f)|$ from algebraic point of view. According to the above facts, it is clear that the upper bound of the number of n-variable Fuzzy/C switching functions is equal to the number of n-variable multiple-valued Kleenean functions whose radix is $|R_0(f)|$.

3. New canonical forms for enumerating Fuzzy/C switching functions

Let $R_0(f)=\{c_0,c_1,\cdots,c_K,c_{K+1},1-c_k,\cdots,1-c_0\}$ be the base set of a Fuzzy/C switching function f (where $c_0=0$, $c_{K+1}=0.5$ and $c_i<c_{i+1}$). The canonical disjunctive (conjunctive) normal form of f is represented as follows:

$$F=F_{P1}\vee F_{P2}\vee F_{P3} \quad (F=F_{C1}\cdot F_{C2}\cdot F_{C3})$$

Where F_{P1}, F_{P2} and F_{P3} (F_{C1}, F_{C2} and F_{C3}) are the disjunction of simple phrases of Type P1 (the conjunction of simple clauses of Type C1), the disjunction of minterms of Type P2 (the conjunction of maxterms of Type C2) and the disjunction of complementary minterms of Type P3 (the conjunction of complementary maxterms of Type C3), respectively.

[Lemma 1] Let f be a Fuzzy/C switching function. For $\exists i,\exists j(\forall A\in f^{-1}(c_i),(\forall B\in f^{-1}(c_j))$, $A\Delta B=\phi$ holds (where the notation $A\Delta B$ means a set of the greatest lower bound of A and B with respect to the partially ordered relation $\prec$.). (Proof is omitted.)

[Theorem 7] Let f be a Fuzzy/C switching function. Then the following two equations hold:

(1) $f=F_{P1}\vee 0.5\cdot F_{C1}$

(2) $f=F_{C1}\cdot(0.5\vee F_{P1})$

(Proof) Subsets of V_3^n which the value of f is less than c_{K+1} $(=0.5)$ are $f^{-1}(c_0),f^{-1}(c_1),\cdots,f^{-1}(c_K)$, respectively. According to theorem 5, maximum elements in these subsets are corresponding to elements which correspond to simple phrases of Type C1 which appear in the canonical conjunctive normal form of f by definition 7. Accordingly, $F_{C1}=F_{C1}^{c_0}\cdot F_{C1}^{c_1}\cdots F_{C1}^{c_K}$ is one of functions whose functional values are $c_0,c_1,\cdots,c_K$ in $f^{-1}(c_0),f^{-1}(c_1),\cdots,f^{-1}(c_K)$, respectively. Similarly, $F_{P1}=F_{P1}^{1-c_0}\vee F_{P1}^{1-c_1}\vee\cdots\vee F_{P1}^{1-c_K}$ is one of functions whose functional values are $1-c_0,1-c_1,\cdots,1-c_K$ in $f^{-1}(1-c_0),f^{-1}(1-c_1),\cdots,f^{-1}(1-c_K)$, respectively. According to lemma 1, for $\forall A\in f^{-1}(c_0)\cup f^{-1}(c_1)\cup\cdots\cup f^{-1}(c_K)$ and $\forall B\in f^{-1}(1-c_0)\cup\cdots\cup f^{-1}(1-c_K)$, $A\Delta B=\phi$ holds. Therefore, $F_{P1}(A)=(F_{P1}^{1-c_0}\vee F_{P1}^{1-c_1}\vee\cdots\vee F_{P1}^{1-c_K})(A)=0$ and $F_{C1}(B)=(F_{C1}^{c_0}\cdot F_{C1}^{c_1}\cdots F_{C1}^{c_K})(B)=1$. Moreover, the value of f in $f^{-1}(c_{K+1})$ must be 0.5. Consequently, $f=F_{P1}\vee 0.5\cdot F_{C1}$ holds. $f=F_{C1}\cdot(0.5\vee F_{P1})$ can be proved in the same manner. //

The logic formulas shown in theorem 7 can be determined uniquely for any Fuzzy/C switching function. Therefore, we can adopt them as new canonical forms of Fuzzy/C switching functions.

In next section, we apply these canonical forms to enumerating the number of Fuzzy/C switching functions.

4. Enumerating Fuzzy/C switching functions.

According to theorem 4, any Fuzzy/C switching function can be represented in the canonical disjunctive normal form. If the number of variables is n, sum of the number of simple phrases of Type P1, the number of minterms of Type P2 and complementary minterms of Type P3 is at most $3^n\cdot(3^n+2^n+(3^n-2^n))=2\cdot 3^{2n}$

Therefore, the number of n-variable Fuzzy/C switching functions is at most $2^{2\cdot 3^{2n}}$. However, it is much fewer in practice because a lot of phrases are subsumed and omitted by other phrases.

Hereafter, $f^{-1}(c_r)=\{A\in V_3^n \mid f(A)=c_r, c_r\in R_0(f)\}$ is called c_r-set.

For n-variavle Fuzzt/C switching function f, if the domain of f is restricted to V_2^n, then each c_r-set is a subset of V_2^n. And it is determined uniquely for any f.

[Definition 6] Let f be a Fuzzy/C switching function. If f is represented in the following logic formula, then f is said to be represented in a *disjunctive form with constants*:

$f=\alpha_1\vee\alpha_2\vee\cdots\vee\alpha_s$

(Where α_i is a simple phrase of Type P1 and $\alpha_i\not\subset\alpha_j$ holds for $i\neq j$.)

Similarly, if f is represented in the following logic formula, then f is said to be represented in a *conjunctive form with constants*:

$f=\beta_1\cdot\beta_2\cdots\beta_t$

(Where β_i is a simple clause of Type C1 and $\beta_i\not\subset\beta_j$ holds $i\neq j$.) //

[Theorem 8] Let f be a Fuzzy/C switching function, and it is represented in a canonical form such as $f=F_{P1}\vee 0.5\cdot F_{C1}$ in theorem 7. Then, for any element of c_r-set in V_2^n, that is, for $\forall A\in f^{-1}(c_r)\cap V_2^n$, F_{P1} is a disjunctive form with constants which satisfies the following condition:

$$F_{P1}(A)=\begin{cases} c_r & (for\ c_r>0.5)\\ 0 & (for\ c_r\le 0.5)\end{cases}$$

Similarly, for any element of c_r-set in V_2^n, F_{C1} is a conjunctive form with constants which satisfies the following condition:

$$F_{C1}(A)=\begin{cases} 1 & (for\ c_r\ge 0.5)\\ c_r & (for\ c_r<0.5)\end{cases}$$

(Proof is omitted.)

Hereafter, the notation (2,p)-function means an n-variable binary input p-valued output switching function.

[Definition 9] Let f, $R_0(f)$ and $|R_0(f)|$ be an n-variable Fuzzy/C switching function, the base set of f and cardinarity of $R_0(f)$, respectively. And let F_{c_r} be an n-variable $(2, |R_0(f)|)$-function (that is, $F_{c_r} : V_2^n \rightarrow R_0(f)$) which has the same $c_r - set$ on V_2^n as f. Then, f and F_{c_r} are said to be $c_r - equivalent$ to each other if $f(A) = F_{c_r}(A)$ holds for any element $(\in f^{-1}(c_r) \cap V_2^n)$. //

In general, there are many n-variable Fuzzy/C switching functions which are c_r-equivalent to F_{c_r}. However, the number of such functions is finite.

Let f be a Fuzzy/C switching function. And let F_P be a (2,p+1)-function (where $p = (|R_0(f)| - 1)/2$) which is c_r-equivalent to f, that is,

$$F_P = \begin{cases} c_r & (for\ \ c_r > 0.5) \\ 0 & (for\ \ c_r \leq 0.5) \end{cases}$$

Similarly, let F_C be a (2,p+1)-function which is c_r -equivalent to f, that is,

$$F_C = \begin{cases} 1 & (for\ \ c_r < 0.5) \\ c_r & (for\ \ c_r \geq 0.5) \end{cases}$$

Here let $DF_{(c_r)}(F_P)$ be the set of all the disjunctive forms with constants which are c_r-equivalent to f, and $|DF_{(c_r)}(F_P)|$ be the cardinality of $DF_{(c_r)}(F_P)$. Similarly, let $CF_{(c_r)}(F_C)$ be the set of all the conjunctive forms with constants and $|CF_{(c_r)}(F_C)|$ be the cardinality of $CF_{(c_r)}(F_C)$.

[Theorem 9] Let f be a Fuzzy/C switching function. And let $c_r - eq(f_{c_r})$ be the set of all the n-variable (2, $|R_0(f)|$)-functions, and $|c_r - eq(f_{c_r})|$ be the cardinality of $c_r - eq(f_{c_r})$. Then, the following equation holds:

$$|c_r - eq(f_{c_r})| = |DF_{(c_r)}(F_P)| \times |CF_{(c_r)}(F_C)|$$

(Proof) According to theorem 7, any Fuzzy/C switching function f can be represented in the canonical form $f = F_{P1} \vee 0.5 \cdot F_{C1}$. Therefore, we can prove this theorem using theorem 8. //

Fuzzy/C switching functions are a model of Kleenean algebra. Therefore, De Morgan low holds for them. Let β_i be a simple clause of Type C1, and $\beta_i = c_i \vee \beta_i^0$ (where $c_i (< 0.5)$ is a constant in $R_0(f)$ and β_i^0 is a conjunction of some literals.). Then, by applying De Morgan low, $\sim F_C = \sim (\underset{i}{\cdot}(c_i \vee \beta_i^0))$ $= \underset{i}{\vee}(\sim c_i \cdot \sim \beta_i^0)$ holds. $\sim c_i \cdot \sim \beta_i^0$ can be represented in a disjunction of simple phrases of Type P1. Consequently, the number of conjunctive forms of F_Cs is equal to the number of disjunctive forms of $\sim F_C$s. Consequently, the following theorem holds:

[Theorem 10] Let f be a Fuzzy/C switching function. Then the following equation holds:

$$|c_r - eq(f_{c_r})| = |DF_{(c_r)}(F_P)| \times |DF_{(c_r)}(\sim F_C)|$$

(Proof is omitted.)

According to theorem 10, we can get the number of n-variable Fuzzy/C switching functions by obtaining the sum of $|c_r - eq(f_{c_r})|$s for $c_r < 0.5$, $c_r = 0.5$ and $c_r > 0.5$, respectively. Consequently, it is shown that the problem for enumerating Fuzzy/C switching functions can result in that of binary switching functions by using a similar method used in previous works[7,8] by the authors.

5. Conclusions

In this paper, we proposed new canonical forms for enumeration and clarified the method to obtain the number of n-variable Fuzzy/C switching functions. However, we have not obtained the concrete number. In future works, we will clarify it and estimate.

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