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Generation of Finite Fuzzy Algebra and Finite De Morgan Algebra Using a Computer

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Abstract

It is well known that a Boolean algebra is one of the most important algebra for engineering. A fuzzy algebra, which is referred to also as a Kleene algebra, is obtained from a Boolean algebra by replacing the complementary law in the axioms of a Boolean algebra with the Kleene's law, where the Kleene's law is a weaker condition than the complementary law. Removal of the Kleene's law from a Kleene algebra gives a De Morgan algebra. In this paper, we generate lattice structures of the above related algebraic systems having finite elements by using a computer. From the result, we could find out a hypothesis that the structure excepting for each element name between a Kleene algebra and a De Morgan algebra is the same from the lattice standpoint.
Keywords: Boolean algebra, fuzzy algebra, Kleene algebra, De Morgan algebra, finite lattice structure

1. Introduction

It is well known that a Boolean algebra is one of the most important algebra for engineering, and the ordinary set theory and the two-valued logic are models or interpretation of a Boolean algebra, respectively.

On the other hand, since L. A. Zadeh introduced the concept of fuzzy set theory in an attempt to treat ambiguity, an infinite multiple-valued logic, which is called a fuzzy logic, whose truth value is continuous between 0 and 1, has been studied. However, many fuzzy logic have ever been proposed[1,2] such as a intuitionistic fuzzy logic, a Lukasiewicz's fuzzy logic, a Kleene's fuzzy logic, and so on. A fuzzy algebra discussed in this paper is the algebraic system which is obtained by abstracting a Zadeh's fuzzy set theory or a Kleene's fuzzy logic. The Zadeh's fuzzy set theory and the Kleene's fuzzy logic are corresponding to each other as good as the ordinary set theory and the two-valued logic, respectively. Hereafter, the Zadeh's fuzzy set theory is called the fuzzy set theory and the Kleene's fuzzy logic is called the Kleene logic for short.

We consider two algebras in this paper. One is a Kleene's fuzzy algebra[3], which is obtained by replacing the complementary low in axiom system of a Boolean algebra with the weaker condition called the Kleene's low. This algebra is also called a soft algebra[4] in the field of engineering, a Kleene algebra[5] in the field of mathematics. This paper uses a Kleene algebra as a substitute for a Kleene's fuzzy algebra. The other is a De Morgan algebra[6] which is obtained by removing the Kleene's low from the axiom system of a Kleene algebra (It is also called a quasi-Boolean algebra[7]).

This paper aims at deriving all the lattice structure of finite fuzzy algebras and finite De Morgan algebras. To clarify the structure of a target algebra, it is important to investigate that if we assume what kind of partially ordered relations for any given finite set, it forms the target one. As an typical example, the representation theorem of a finite Boolean algebra whose the number of elements is n, that is, the lattice structure of a Boolean algebra is isomorphic to that of lattice of set called 2^n, is well known. However, representation theorems of a finite fuzzy algebra and a finite De Morgan algebra have never been known. Using a computer, the authors generated all the lattice structure each of which forms a finite fuzzy algebra and/or a finite De Morgan algebra for any given poset whose the number of elements is thirteen or less than. Accordingly, from lattice structure's point of view, we could find out an interesting result, which the lattice structure of sets forming a finite fuzzy algebra and a finite De Morgan algebra are the same each other. The condition for which there exist fixed points is one of the typical difference between a fuzzy algebra and a De Morgan algebra, and has ever been studied. According to our result, it would be expected that the above two algebras are isomorphic to each other if we remove the name of elements which are fixed points. This expectation means that we would be able to develop the representation theorem of a finite De Morgan algebra by adding the condition for which there exist fixed points to that of a finite fuzzy algebra.

2. Preliminaries

Let a, b, c be any elements of a non-empty set L. A lattice $\langle L, \vee, \wedge \rangle$ is an algebraic system in which the two operations $\vee$ and $\wedge$ are defined on the set L and which satisfies the following six axioms.

(1) the commutative law

(a) $a \vee b = b \vee a$, (b) $a \wedge b = b \wedge a$,

(2) the associative law

(a) $a \vee (b \vee c) = (a \vee b) \vee c$,

(b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,

(3) the absorption law

(a) $a \vee (a \wedge b) = a$, (b) $a \wedge (a \vee b) = a$.

In particular, if a lattice satisfies

(*) the modular law

$a \geq c \rightarrow a \wedge (b \vee c) = (a \wedge b) \vee c$,

then it is called a modular lattice. If the modular law is replaced in a modular lattice with

(4) the distributive law

(a) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(b) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,

which are strong conditions than the modular law, then we have a distributive lattice. In a distributive lattice, if an unary operation N is defined and, in addition, it satisfies

(5) De Morgan's law

(a) $N(a \vee b) = Na \wedge Nb$,

(b) $N(a \wedge b) = Na \vee Nb$,

(6) the double negation law

$N(Na) = a$,

then the lattice $\langle L, \vee, \wedge, N \rangle$ is said to be a De Morgan lattice. A De Morgan algebra[6] is an algebraic system $\langle L, \vee, \wedge, N, 0, 1 \rangle$ which is a De Morgan lattice and has the least element 0 and the greatest element 1 satisfying

(7) the least element

(a) $0 \vee a = a$, (b) $0 \wedge a = 0$,

(8) the greatest element

(a) $1 \vee a = 1$, (b) $1 \wedge a = a$.

A Kleene algebra[3,5] is a De Morgan algebra and, furthermore, satisfies

(9) Kleene's law

(a) $(a \wedge Na) \vee (b \vee Nb) = b \vee Nb$,

(b) $(a \wedge Na) \wedge (b \vee Nb) = a \wedge Na$.

If we replace the Kleene's law of a Kleene algebra with stronger conditions

(**) the complementary law

(a) $a \vee Na = 1$, (b) $a \wedge Na = 0$,

then we have a Boolean algebra.

In an algebraic system, many equalities are valid. Among these equalities, some are derived from others, that is, they are not independent of each other. Here, the following problem arises: which equalities can we adopt as axioms from which all remaining equalities are derivable? A set of axioms from which all valid equalities in the algebraic system are derived is called a complete set of axioms for the algebraic system. Furthermore, if the axioms are independent of each other, that is, each axiom can not be derived from others, then the set of axioms is said to be independent. In case of examining whether any given finite set satisfies the algebraic system, it is sufficient to examine the complete and independent axioms for the algebraic system. As for the equalities from (1) to (9) mentioned above, the axioms shown in the first half of the following definition 1 are a set of complete and independent axioms for a Kleene algebra[3], and furthermore, the axioms shown in the latter half are a set of complete and independent axioms for a De Morgan algebra[3].

[Definition 1] The six axioms from (K1) to (K6) mentioned below represent one example of complete and independent axioms for a Kleene algebra.

(K1) the commutative law

(a) $a \vee b = b \vee a$,

(K2) the distributive law

(a) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(K3) De Morgan's law

(a) $N(a \vee b) = Na \wedge Nb$,

(K4) the double negation law

$N(Na) = a$,

(K5) the least element

(a) $0 \vee a = a$,

(K6) Kleene's law

(a) $(a \wedge Na) \vee (b \vee Nb) = b \vee Nb$.

If the axiom (K6) in the above is replaced with

(D6) the least element

(b) $0 \wedge a = 0$,

then the six axioms which satisfy from (K1) to (K5) and (D6) represent one example of complete and independent axioms for a De Morgan algebra. □

Using the complete and independent axioms of an algebraic system in this paper is effective for the following reasons: we can clarify the structure of algebraic system by examining the role of each axioms, and we can know whether any given model satisfies the algebraic system by simply examining each axioms.

3. Generation of finite algebra

The enumeration of finite lattices having n elements has been reported in the case of $n \leq 11$ by Tamura and Tanaka[8], whose method is the first one as far as we know for generating finite lattices constructively by using a computer. It is necessary to derive the finite lattice before to generate any finite algebra in this paper. On this subject, we adopt the method proposed by Tamura and Tanaka[8], which is as follows.

Let P be a set of n elements, which is denoted by $P = \{p_1, p_2, \cdots, p_n\}$. Since any finite lattice always contains the greatest element and the least element, we can decide previously p_1 as the greatest element and p_n as the least element respectively. To describe a partially ordered relation between any two elements p_i and p_j (where, $1 \leq i, j \leq n$) in P, the $n \times n$ matrix which is denoted by $R = (r_{ij})$ is defined as follows.

$$r_{ij} = \begin{cases} 1 & (\ p_i \geq p_j,\ \text{where}\ p_i \neq p_j\ ) \\ -1 & (\ p_i \leq p_j,\ \text{where}\ p_i \neq p_j\ ) \\ 0 & (\ \text{the other}\ ) \end{cases}$$

This matrix is called the relationship matrix for P in this paper. By means of a proper permutation among the elements $p_1, p_2, \cdots, p_n$ in P, the elements r_{ij} of a matrix $R = (r_{ij})$, which are about a half of elements in R and are shaped a triangle on the upper right from the diagonal elements of R (that is, $i \le j$ holds for any element r_{ij}), have made possible to exclude -1 as the value. On the other hand in this case, the elements r_{ij} which are shaped a triangle on the lower left from the diagonal elements of R (that is, $i \ge j$ holds for any element r_{ij}) can not take 1 as the value. Furthermore, since the element p_1 is determined the greatest element previously, the relation $r_{1j} = 1$ always holds for $2 \le j \le n$ (naturally, $r_{i1} = -1$ holds for $2 \le i \le n$). Similarly, since the element p_n is determined the least element previously, the relation $r_{in} = 1$ always holds for $1 \le i \le n-1$ (naturally, $r_{nj} = -1$ holds for $1 \le j \le n-1$). Then, the relationship matrix R is represented by the following style.

$$R = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & 0 & \alpha & \vdots \\ \vdots & \beta & \ddots & 1 \\ -1 & \cdots & -1 & 0 \end{pmatrix}$$

Here, any element of the part denoted as α in R takes either 0 or 1, and any element of the part denoted as β in R takes either 0 or -1. By the way, in the case of satisfying the commutative law (a) in a given axioms, the following two conditions hold;

if $r_{ij} = 1$ then $r_{ji} = -1$, and if $r_{ij} = 0$ then $r_{ji} = 0$. Namely, if a relation r_{ij} of α in R takes either 0 or 1, then the relation r_{ji} of β in R, which is exchanged between the row and the column symmetrically for the relation r_{ij}, takes either 0 or -1, respectively. Then, the relationship matrix R is uniquely determined except for the part denoted by β. Consequently, the relationship matrix R can be uniquely represented by the binary representation through the arrangement according to elements r_{ij} of α as follows.

$$r_{23},\ r_{24}r_{34},\ \cdots,\ r_{2k}r_{3k}\cdots r_{(k-1)k},\ \cdots,\ r_{2(n-1)}r_{3(n-1)}\cdots r_{(n-2)(n-1)}$$

For any given finite set P, the set P must at least be a partially ordered set in order to be a lattice. In the case of examining whether the set P satisfies a partially ordered set, the binary representation mentioned above must be satisfied the following three properties.

reflexive: $(p_i \le p_i)$,

antisymmetric: $(p_i \le p_j \ \&\ p_j \le p_i \ \rightarrow\ p_i = p_j)$,

transitive: $(p_i \le p_j \ \&\ p_j \le p_k \ \rightarrow\ p_i \le p_k)$.

Here, since the binary representation has been made satisfying both the reflexive property and the antisymmetric property, it is sufficient to examine only the transitive property. This is judged as follows.

For $i < j < k$, if $r_{ij} = 1$ and $r_{jk} = 1$ hold, then $r_{ik} = 1$ holds (this condition is the same as follows; if $r_{ij} = 1$ and $r_{ik} \ne 1$ hold, then $r_{jk} \ne 1$ holds). Furthermore, in order to judge the set P to be a lattice, for any two elements p_i and p_j (where, $i \ne j$) in P, we must examine whether both $p_i \vee p_j$ and $p_i \wedge p_j$ uniquely exist as the elements of P. Here, if $p_i \vee p_j$ exists, then $p_i \wedge p_j$ always exists. Because, if we take P_L to a subset of P such as

$$P_L = \{\, l \mid l \le p_i \text{ and } l \le p_j \,\},$$

then P_L is not empty since the least element is included in P_L, and the infimum element $p_i \wedge p_j$ is uniquely determined as follows.

$$\bigvee_{l \in P_L} l = p_i \wedge p_j$$

The judgement which $p_i \vee p_j$ uniquely exists is as follows. For any i, j (where, $i < j$) such as to hold $r_{ij} = 0$, let $p_{u_1}, p_{u_2}, \cdots, p_{u_k}$ be the upper bounds of $\{p_i, p_j\}$ in P. If it may take a number m to be the maximum one among the subscripts $u_1, u_2, \cdots, u_k$, then the element p_{u_t}, which satisfies $p_{u_t} \ge p_m$ for any t where $1 \le t \le k$ holds, can be uniquely determined as the supremum element $p_i \vee p_j$.

The judging method for a finite lattice is as mentioned above, then all finite lattices each of which has both the greatest element and the least element can be generated by the binary representations. In reference [8], the binary representation, which is obtained as the minimum one by permuting each other among the elements $p_2, \cdots, p_{n-1}$ in P, is called the expression for a finite lattice, and is designed that it may avoid the duplication among some finite lattices which are isomorphic with one another. Table 1 shows the generating results of several algebraic systems which are obtained in this paper, such as a finite lattice (Lat,

Table 1 The number of several algebraic systems (where, the notation n indicates the number of elements)

n	Lat	Mod	D	M	K	B
3	1	1	1	1	1	-
4	2	2	2	2	2	1
5	5	4	3	1	1	-
6	15	8	5	3	3	-
7	53	16	8	2	2	-
8	222	34	15	6	6	1
9	1078	72	26	4	4	-
10	5994	157	47	10	10	-
11	37622	343	82	6	6	-

whose notation in the parenthesis denotes a target algebraic system in the table 1, and the notations mentioned below are in the same way), a finite modular lattice (Mod), a finite distributive lattice (D), where these finite lattices have always the greatest element and the least element, and furthermore, a finite De

Morgan algebra (M), a finite Kleene algebra (K) and a finite Boolean algebra (B), respectively. This table is obtained by both using the method proposed by Tamura and Tanaka[8] and individually increasing the axioms described in section 2.

The binary representations, each of which is obtained as a finite modular lattice (Mod) having n elements where $n \leq 8$ holds, are as follows.

Number of elements = 3

No. 1		D, M, K

Number of elements = 4

No. 1	0	D, M, K, B
No. 2	1	D, M, K

Number of elements = 5

No. 1	0, 00	
No. 2	0, 11	D
No. 3	1, 10	D
No. 4	1, 11	D, M, K

Number of elements = 6

No. 1	0, 00, 000	
No. 2	0, 00, 111	
No. 3	0, 01, 110	D, $\underline{M}$, $\underline{K}$
No. 4	0, 11, 111	D.
No. 5	1, 10, 100	
No. 6	1, 10, 111	D, M, K
No. 7	1, 11, 110	D
No. 8	1, 11, 111	D, M, K

Number of elements = 7

No. 1	0, 00, 000, 0000	
No. 2	0, 00, 000, 1111	
No. 3	0, 00, 001, 1110	
No. 4	0, 00, 111, 1111	
No. 5	0, 01, 010, 1100	
No. 6	0, 01, 110, 1111	D
No. 7	0, 11, 111, 1110	D, M, K
No. 8	0, 11, 111, 1111	D
No. 9	1, 10, 100, 1000	
No. 10	1, 10, 100, 1111	
No. 11	1, 10, 101, 1110	D
No. 12	1, 10, 111, 1111	D
No. 13	1, 11, 110, 1100	
No. 14	1, 11, 110, 1111	D
No. 15	1, 11, 111, 1110	D
No. 16	1, 11, 111, 1111	D, M, K

Number of elements = 8

No. 1	0, 00, 000, 0000, 00000	
No. 2	0, 00, 000, 0000, 11111	
No. 3	0, 00, 000, 0001, 11110	
No. 4	0, 00, 000, 1111, 11111	
No. 5	0, 00, 001, 0010, 11100	
No. 6	0, 00, 001, 1110, 11111	
No. 7	0, 00, 011, 1010, 11000	D, M, $\underline{K}$, $\underline{B}$
No. 8	0, 00, 111, 1111, 11110	
No. 9	0, 00, 111, 1111, 11111	
No. 10	0, 01, 010, 0100, 11000	
No. 11	0, 01, 010, 1100, 11111	
No. 12	0, 01, 011, 1100, 11101	D, $\underline{M}$, $\underline{K}$
No. 13	0, 01, 110, 1101, 11110	D
No. 14	0, 01, 110, 1111, 11111	D
No. 15	0, 11, 111, 1110, 11100	
No. 16	0, 11, 111, 1110, 11111	D
No. 17	0, 11, 111, 1111, 11110	D, M, K
No. 18	0, 11, 111, 1111, 11111	D
No. 19	1, 10, 100, 1000, 10000	
No. 20	1, 10, 100, 1000, 11111	
No. 21	1, 10, 100, 1001, 11110	
No. 22	1, 10, 100, 1111, 11111	
No. 23	1, 10, 101, 1010, 11100	
No. 24	1, 10, 101, 1110, 11111	D, $\underline{M}$, $\underline{K}$
No. 25	1, 10, 111, 1111, 11110	D
No. 26	1, 10, 111, 1111, 11111	D
No. 27	1, 11, 110, 1100, 11000	
No. 28	1, 11, 110, 1100, 11111	
No. 29	1, 11, 110, 1101, 11110	D
No. 30	1, 11, 110, 1111, 11111	D, M, K
No. 31	1, 11, 111, 1110, 11100	
No. 32	1, 11, 111, 1110, 11111	D
No. 33	1, 11, 111, 1111, 11110	D
No. 34	1, 11, 111, 1111, 11111	D, M, K

As for the binary representations each of which forms a finite modular lattice (Mod) mentioned above, the notations such as D, M, K and B indicate that each one also satisfies the axioms of a algebraic system such as a distributive lattice having the greatest element and the least element, a De Morgan algebra, a Kleene algebra and a Boolean algebra, respectively. And the underline for each notation means that the finite algebra which is indicated the notation is isomorphic to the corresponding binary representation as a lattice-ordered set. From this result, it is expected that the structure between a finite Kleene algebra and a finite De Morgan algebra is the same except for each element name.

4. Kleene algebra and De Morgan algebra

As it seems that there is a close relation between a finite Kleene algebra and a finite De Morgan algebra from the previous section, we investigate on this point.

With regard to a finite Kleene algebra and a finite De Morgan algebra, some simple properties are applicable about the operator N as follows. Though these are trivial, it is necessary to simplify the decision procedure as mentioned later. The property 1 and the property 2 as follows are satisfied on both a finite Kleene algebra and a finite De Morgan algebra.

[Property 1] If $a \neq b$ holds, then $Na \neq Nb$ holds. (Proof is omitted)

[Property 2] If $Na = b$ holds, then $Nb = a$ holds. (Proof is omitted)

The next property 3 is satisfied only a finite Kleene algebra.

[Property 3] If there exists the element such as $Na = a$, then it is unique. (Proof is omitted)

As for a finite Kleene algebra, the element such as $Na = a$ is called the fixed point[3,9,10,11], and it is uniquely determined if this element exists (, as to a finite De Morgan algebra, there exists some fixed points in general since the property 3 does not hold). From the above properties, except for the fixed point, any element included in either a Kleene algebra or a De Morgan algebra is symmetrically related to the element which is negated itself (that is, the element which is performed an operator N). Then, concerning the operator N, the following equation holds for any element p_i of P which is denoted in section 2.

$$\forall p_i \in P \text{ (where, } 1 \le i \le n \text{)} ; \; Np_i = p_{n-i+1}.$$

In the case of deciding whether any given finite set satisfies the target algebraic system, for example either a Kleene algebra or a De Morgan algebra, we examine only a complete and independent axioms for each algebraic system. Here, if we use the properties as mentioned above, it is possible to make simply the decision procedure as follows.

[Procedure 1] (for a finite Kleene algebra)

(1) generating the binary representation for a relationship matrix R,

(2) deciding whether the binary representation obtained by (1) forms a lattice (the method proposed by Tamura and Tanaka[8]),

(3) examining the distributive law (a) which is transformed as follows,

$$N(Na \vee N(b \vee c)) = N(Na \vee Nb) \vee N(Na \vee Nc)$$

(4) examining the Kleene's law (a) which is transformed as follows,

$$N(Na \vee a) \vee (b \vee Nb) = b \vee Nb$$

(5) deciding whether the binary representation which satisfies the step (3) and the step (4) is isomorphic to the one which is already recorded as a target algebra (if it does not exist, then we record this). □

[Procedure 2] (for a finite De Morgan algebra)

It is the same as that described for the procedure 1 except for the step (4). □

According to the procedure 1 and the procedure 2, we can generate finite Kleene algebras and finite De Morgan algebras respectively, both of which have more elements compared to the result described in section 2. The binary representations, each of which forms a finite Kleene algebra (K) having n elements where $n \le 13$ holds, are as follows.

number of elements = 3
No. 1 M

number of elements = 4
No. 1 0 M
No. 2 1 M

number of elements = 5
No. 1 1, 11 M

number of elements = 6
No. 1 0, 10, 110 M
No. 2 1, 10, 111 M
No. 3 1, 11, 111 M

number of elements = 7
No. 1 0, 11, 111, 1110 M
No. 2 1, 11, 111, 1111 M

number of elements = 8
No. 1 0, 00, 110, 1010, 01100 $\underline{M}$
No. 2 0, 01, 110, 1111, 01100 $\overline{M}$
No. 3 0, 11, 111, 1111, 11110 M
No. 4 1, 10, 110, 1110, 11111 M
No. 5 1, 11, 110, 1111, 11111 M
No. 6 1, 11, 111, 1111, 11111 M

number of elements = 9
No. 1 0, 01, 110, 1000, 11011, 111100 M
No. 2 0, 11, 111, 1111, 11111, 111110 M
No. 3 1, 10, 111, 1111, 11110, 111111 M
No. 4 1, 11, 111, 1111, 11111, 111111 M

number of elements = 10
No. 1 0, 01, 011, 1100, 11101, 111111, 0111000 M
No. 2 0, 01, 110, 1111, 11010, 111111, 1111100 M
No. 3 0, 11, 111, 1110, 11111, 111111, 1111110 M
No. 4 0, 11, 111, 1111, 11111, 111111, 1111110 M
No. 5 1, 10, 100, 1110, 11010, 101100, 1111111 $\underline{M}$
No. 6 1, 10, 101, 1110, 11111, 101100, 1111111 $\overline{M}$
No. 7 1, 10, 111, 1111, 11111, 111110, 1111111 M
No. 8 1, 11, 110, 1110, 11110, 111111, 1111111 M
No. 9 1, 11, 111, 1110, 11111, 111111, 1111111 M
No. 10 1, 11, 111, 1111, 11111, 111111, 1111111 M

number of elements = 11
No. 1 0, 01, 110, 1111, 11111, 111110, 1111111, 11111100 M
No. 2 0, 11, 111, 1111, 11111, 111111, 1111111, 11111110 M
No. 3 1, 10, 101, 1110, 11000, 111011, 1111100, 11111111 M
No. 4 1, 10, 111, 1111, 11111, 111111, 1111110, 11111111 M
No. 5 1, 11, 110, 1111, 11111, 111110, 1111111, 11111111 M
No. 6 1, 11, 111, 1111, 11111, 111111, 1111111, 11111111 M

number of elements = 12
No. 1 0, 00, 001, 0110, 10100, 110000, 1110111, 10110100, 011110000 $\underline{M}$
No. 2 0, 01, 010, 0111, 11000, 111001, 1101010, 11111111, 011110000 $\underline{M}$
No. 3 0, 01, 011, 0111, 11000, 111001, 1111011, 11111111, 011110000 M
No. 4 0, 01, 011, 1100, 11101, 100000, 1100101, 11101111, 111111000 M
No. 5 0, 01, 110, 1101, 11110, 111111, 1101100, 11111111, 111111100 M
No. 6 0, 01, 110, 1111, 11111, 111111, 1111110, 11111111, 111111100 M
No. 7 0, 11, 111, 1110, 11110, 111110, 1111111, 11111111, 111111110 M
No. 8 0, 11, 111, 1111, 11110, 111111, 1111111, 11111111, 111111110 M
No. 9 0, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111110 M
No. 10 1, 10, 101, 1011, 11100, 111101, 1111111, 10111000, 111111111 M
No. 11 1, 10, 101, 1110, 11111, 111010, 1111111, 11111100, 111111111 M
No. 12 1, 10, 111, 1111, 11110, 111111, 1111111, 11111110, 111111111 M
No. 13 1, 10, 111, 1111, 11111, 111111, 1111111, 11111110, 111111111 M
No. 14 1, 11, 110, 1100, 11110, 111010, 1101100, 11111111, 111111111 $\underline{M}$
No. 15 1, 11, 110, 1101, 11110, 111111, 1101100, 11111111, 111111111 M
No. 16 1, 11, 110, 1111, 11111, 111111, 1111110, 11111111, 111111111 M
No. 17 1, 11, 111, 1110, 11110, 111110, 1111111,

	11111111, 111111111	M
No. 18	1, 11, 111, 1111, 11110, 111111, 1111111, 11111111, 111111111	M
No. 19	1, 11, 111, 1111, 11111, 111111, 1111111. 11111111, 111111111	M

number of elements = 13

No. 1	0, 01, 011, 1100, 11101, 111111, 1100100, 11101101, 111111111, 1111111000	M
No. 2	0, 01, 110, 1111, 11111, 111111, 1111111, 11111110, 111111111, 1111111100	M
No. 3	0, 11, 111, 1110, 11111, 111111, 1111110, 11111111, 111111111, 1111111110	M
No. 4	0, 11, 111, 1111, 11111, 111111, 1111111. 11111111. 111111111. 1111111110	M
No. 5	1, 10, 101, 1110, 11111, 111111, 1111110, 11111111, 111111100, 1111111111	M
No. 6	1, 10, 111, 1111, 11111, 111111, 1111111, 11111111, 111111110, 1111111111	M
No. 7	1, 11, 110, 1101, 11110, 111000, 1111011, 11111100, 111111111, 1111111111	M
No. 8	1, 11, 110, 1111, 11111, 111111, 1111111, 11111110, 111111111, 1111111111	M
No. 9	1, 11, 111, 1110, 11111, 111111, 1111110, 11111111, 111111111, 1111111111	M
No. 10	1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111, 1111111111	M

As for the above binary representations each of which satisfies a finite Kleene algebra (K), the notation M indicates a finite De Morgan algebra which is uniquely determined the naming for a element. On the other hand, concerning the naming for a element, the underline for notation M indicates that there always exist some finite De Morgan algebras each of which is isomorphic to the Kleene algebra denoted by a binary representation. From the point of view on a lattice-ordered set except for each element name, it is seem that a finite Kleene algebra and a finite De Morgan algebra are isomorphic to each other. Furthermore, it would be expected that a finite Kleene algebra can be uniquely determined by the binary representation itself (namely, the naming for a element in a target algebra is unique), and a finite De Morgan algebra is not always uniquely determined by the same binary representation as a finite Kleene algebra. The reason is that the condition for which a fixed point exists is not uniquely determined in a De Morgan algebra (thereby, some fixed points exist in general).

5. Conclusion

Nakajima and Morioka[10] have been studied a De Morgan algebra by using a concept called the fixed core which extends the fixed point. And, they proposed two important facts that the fixed core which does not reduce the fixed point forms a Boolean algebra, and a De Morgan algebra which has a unique fixed core forms a Kleene algebra. From our results mentioned above, each of which has n elements where $n \leq 13$ holds, it is seem that the following facts hold: Any finite De Morgan algebra which is isomorphic to the another one always includes a Boolean algebra as a subset of the lattice-ordered set, and if it does not include, then the finite De Morgan algebra forms the same structure as a finite Kleene algebra from the lattice standpoint. These facts have not been proved as far as we know, proving the hypotheses are left unsettled.

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