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AN ACHIEVEMENT RATE APPROACH TO LINEAR PROGRAMMING PROBLEMS WITH CONVEX POLYHEDRAL OBJECTIVE COEFFICIENTS

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Abstract

In this paper, an LP problem with convex polyhedral objective coefficients is treated. In the problem, the interactivities of the uncertain objective coefficients are represented by a bounded convex polyhedron (a convex polytope). We develop a computation algorithm of a maximin achievement rate solution. To solve the problem, first, we introduce the relaxation procedure. In the algorithm, a sub-problem, a bilevel programming problem, should be solved. To solve the sub-problem, we develop a solution method based on a branch and bound method. As a result, it is shown that the problem can be solved by the repetitional use of the simplex method.

Keywords: Possibilistic programming, maximin problem, relaxation procedure, bilevel programming, branch and bound method

1. Introduction

In linear programming (LP) problems with uncertain objective coefficients, robust solutions which minimize the worst deviation from the optimal value have been discussed (see [1] [2]). Especially for interval objective coefficients, solution algorithms for the minimax regret solution [1] and for the maximin achievement rate solution [2] have been proposed. Those algorithms utilize the non-interactivities of the uncertain objective coefficients as the traditional fuzzy and interval programming techniques have utilized. The interactivities among uncertain objective coefficients have not yet been investigated considerably.

In this paper, an LP problem with convex polyhedral objective coefficients is treated. In the problem, the interactivities of the uncertain objective coefficients are represented by a bounded convex polyhedron (a convex polytope). After a review of the traditional approach to the LP problem, an example which makes the approach controversial is presented. The most reasonable solution set, which is called a necessarily optimal solution set, to the LP problem is described. Since the necessarily optimal solution set is empty in many cases, the maximin achievement rate solution is introduced as a solution which minimizes a deviation from the necessary optimality. A computation method of a maximin achievement rate solution is discussed. The maximin achievement rate problem is a maximin problem and a relaxation procedure is applied. The difficulty still remains in solving the sub-problem, which is a bilevel programming problem. To tackle this sub-problem, an equivalent problem is shown and a branch and bound method is applied. As the result, it is shown that a maximin achievement rate solution can be obtained by the repetitional use of the simplex method.

2. Problem and the best solution

The following LP problem with uncertain objective coefficients is treated in this paper.

$$\begin{array}{ll} \text{maximize} & \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}, \\ \text{sub. to} & \boldsymbol{x} \in X = \{\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}, \end{array} \tag{1}$$

where A is an $m \times n$ matrix, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)^{\mathrm{T}}$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_m)^{\mathrm{T}}$. $\boldsymbol{\gamma}$ is not known precisely but the possible range is given as the following non-empty bounded convex polyhedron Γ:

$$\Gamma = \{\boldsymbol{c} = (c_1, c_2, \ldots, c_n)^{\mathrm{T}} \mid D\boldsymbol{c} \leq \boldsymbol{g}\}, \tag{2}$$

where D is a $p \times n$ matrix and $\boldsymbol{g} = (g_1, g_2, \ldots, g_p)^{\mathrm{T}}$. We assume that X is bounded and for any $\boldsymbol{c} \in \Gamma$, $\max_{\boldsymbol{x} \in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} > 0$.

In such an LP problem with uncertain objective coefficients, the objective function value of a feasible solution $\boldsymbol{x}$ cannot be obtained exactly but the range is obtained as an interval,

$$\left[\min_{\boldsymbol{c} \in \Gamma} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x},\ \max_{\boldsymbol{c} \in \Gamma} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}\right]. \tag{3}$$

Since the interval objective function value with the maximal lower and upper bounds is preferable, (1) has often treated by a biobjective programming

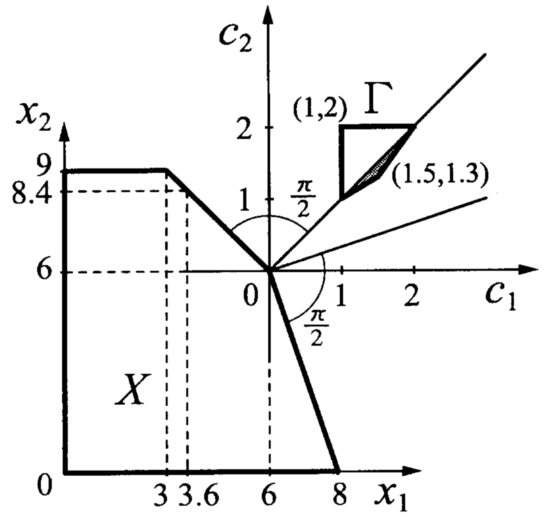

Fig. 1 An example

problem (see [3] [4]),

$$\begin{aligned} \text{maximize} \quad & \min_{\boldsymbol{c} \in \Gamma} \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}, \\ \text{maximize} \quad & \max_{\boldsymbol{c} \in \Gamma} \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}, \\ \text{sub. to} \quad & \boldsymbol{x} \in X. \end{aligned} \tag{4}$$

A completely optimal solution to (4), which maximizes the both objective functions at the same time, has been regarded as the best solution.

However, as exemplified in Example 1, a complete optimal solution is not always the best solution.

Example 1. Let us consider the following LP problem with convex polyhedral objective coefficients:

$$\begin{aligned} \text{maximize} \quad & \gamma_1 x_1 + \gamma_2 x_2, \\ \text{sub. to} \quad & x_1 + x_2 \le 12, \\ & 3x_1 + x_2 \le 24, \\ & 0 \le x_2 \le 9, \; x_1 \ge 0, \end{aligned} \tag{5}$$

where $\gamma = (\gamma_1, \gamma_2)^{\mathrm{T}}$ is restricted by

$$\begin{aligned} \Gamma = \{(c_1, c_2)^{\mathrm{T}} \mid 7c_1 - 5c_2 \le 4, \; c_2 \le 2, \\ -3c_1 + 5c_2 \ge 2, \; c_1 \ge 1\}. \end{aligned} \tag{6}$$

For every $\boldsymbol{c} \in \Gamma$, we have $(1,1)^{\mathrm{T}} \le \boldsymbol{c} \le (2,2)^{\mathrm{T}}$, $(1,1)^{\mathrm{T}} \in \Gamma$ and $(2,2)^{\mathrm{T}} \in \Gamma$. Thus, the biobjective programming problem becomes

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2, \\ \text{maximize} \quad & 2x_1 + 2x_2, \\ \text{sub. to} \quad & \boldsymbol{x} = (x_1, x_2)^{\mathrm{T}} \in X. \end{aligned} \tag{7}$$

This problem has a completely optimal solution $\boldsymbol{x}^* = (6,6)^{\mathrm{T}}$.

The solution is illustrated in Fig. 1. The shaded area of Fig. 1 is the set of $\boldsymbol{c} \in \Gamma$ which makes $(6,6)^{\mathrm{T}}$ optimal. Despite the complete optimality, this shaded area is small relatively to Γ. From this point of view, $(6,6)^{\mathrm{T}}$ is not necessarily the best solution. □

As exemplified by Example 1, a completely optimal solution to (4) is not always the best solution. The best solution should be included in the following set:

$$NS = \bigcap_{\boldsymbol{c} \in \Gamma} S(\boldsymbol{c}), \tag{8}$$

where $S(\boldsymbol{c})$ is defined by

$$S(\boldsymbol{c}) = \left\{ \boldsymbol{y} \in X \;\middle|\; \boldsymbol{c}^{\mathrm{T}} \boldsymbol{y} = \max_{\boldsymbol{x} \in X} \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} \right\}. \tag{9}$$

NS is called 'a necessarily optimal solution set' (see [5]). An element of NS is a solution which is optimal for all $\boldsymbol{c} \in \Gamma$.

In many cases, NS is empty. A solution which is a necessarily optimal solution when NS is nonempty, and otherwise, which minimizes a deviation from the necessary optimality must be preferable. As such solutions, a minimax regret solution [1] and a maximin achievement rate solution [2] are proposed. In this paper, a solution algorithm of a maximin achievement rate solution is discussed.

3. Maximin achievement rate solution

3.1. Solution concept

Assume we know the true objective function coefficient vector $\boldsymbol{c}$ after the determination of the solution of (1) as $\boldsymbol{x}$. Under this assumption, the ratio of the obtained objective function value $\boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}$ to the optimal value is given by

$$ra(\boldsymbol{x}, \boldsymbol{c}) = \frac{\boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}}{\max_{\boldsymbol{y} \in X} \boldsymbol{c}^{\mathrm{T}} \boldsymbol{y}}. \tag{10}$$

From the assumption of (1), we have $ra(\boldsymbol{x}, \boldsymbol{c}) \le 1$. Thus, the larger $ra(\boldsymbol{x}, \boldsymbol{c})$ is, the better the solution is. We can regard $ra(\boldsymbol{x}, \boldsymbol{c})$ as the achievement rate.

At the decision stage, $\boldsymbol{c}$ is unknown. The worst (minimum) achievement rate can be defined by

$$Ra(\boldsymbol{x}) = \min_{\boldsymbol{c} \in \Gamma} ra(\boldsymbol{x}, \boldsymbol{c}). \tag{11}$$

Thus, (1) can be formulated as a maximization problem of $Ra(\boldsymbol{x})$, i.e.,

$$\underset{\boldsymbol{x} \in X}{\text{maximize}} \; \min_{\boldsymbol{c} \in \Gamma} \frac{\boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}}{\max_{\boldsymbol{y} \in X} \boldsymbol{c}^{\mathrm{T}} \boldsymbol{y}}. \tag{12}$$

The solution to this problem called 'a maximin achievement rate solution'.

When the optimal value of (12) is 1, a maximin achievement rate solution is a necessarily optimal solution, and otherwise, a maximin achievement rate solution minimizes the deviation from the necessary optimality defined by $1 - Ra(\boldsymbol{x})$.

3.2. Relaxation approach

Since (12) is a maximin problem, we can apply a relaxation procedure (see [6]). Introducing the relaxation procedure, the solution to (12) can be obtained by the following algorithm with an admissible error $\varepsilon > 0$.

[Algorithm 1]

Step 1 Obtain a feasible solution $\boldsymbol{x}^0 \in X$. Set $k = 1$ and $r^0 = 1$.

Step 2 Solve the sub-problem to obtain $Ra(\boldsymbol{x}^0)$. Let $(\boldsymbol{c}^k, \ \boldsymbol{y}^k)$ be an optimal solution.

Step 3 If $Ra(\boldsymbol{x}^0) \geq r^0 - \varepsilon$, then terminate. In this case, $\boldsymbol{x}^0$ is the solution with admissible error ε and r^0 is the minimum achievement.

Step 4 Solve an LP problem,

$$\begin{aligned} &\text{maximize} \quad r \\ &\text{sub. to} \quad A\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq 0, \\ &\qquad\qquad \frac{(\boldsymbol{c}^j)^{\mathrm{T}}\boldsymbol{x}}{(\boldsymbol{c}^j)^{\mathrm{T}}\boldsymbol{y}^j} \geq r, \ j = 1, 2, \ldots, k. \end{aligned} \tag{13}$$

Update $(\boldsymbol{x}^0, r^0)$ with an optimal solution to (13). $k = k + 1$. Return to Step 2.

Since X is a closed bounded set, this algorithm terminates in a finite iterations.

There still exists a difficulty to solve the sub-problem at Step 2. In what follows, we discuss a solution algorithm for the sub-problem.

3.3. Solving the sub-problem

The sub-problem can be rewritten as,

$$\begin{aligned} &\underset{\boldsymbol{c},\,\boldsymbol{y}}{\text{minimize}} \quad \frac{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^0}{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{y}}, \\ &\text{sub. to} \quad D\boldsymbol{c} \leq \boldsymbol{g}, \ A\boldsymbol{y} = \boldsymbol{b}, \ \boldsymbol{y} \geq 0, \\ &\qquad\qquad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} = \max_{\boldsymbol{z}} \ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{z}, \\ &\qquad\qquad\qquad\qquad \text{sub. to} \quad A\boldsymbol{z} = \boldsymbol{b}, \ \boldsymbol{z} \geq 0. \end{aligned} \tag{14}$$

This kind of problem is famous as a bilevel programming problem or a Stackelberg problem (see [7]). Some solution algorithms have been proposed to solve a bilevel programming problem under some assumptions such as the convexity, the differentiability and so forth. However, unfortunately, (14) does not satisfy those assumptions because of the non-convexity of the objective function. We propose a solution method for (14).

Applying the optimality condition of an LP problem (see, for example, [8]), (14) becomes

$$\begin{aligned} &\underset{\boldsymbol{c},\,\boldsymbol{y},\,\boldsymbol{u}}{\text{minimize}} \quad \frac{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^0}{\boldsymbol{b}^{\mathrm{T}}\boldsymbol{u}}, \\ &\text{sub. to} \quad D\boldsymbol{c} \leq \boldsymbol{g}, \ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{u} = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y}, \ A^{\mathrm{T}}\boldsymbol{u} \geq \boldsymbol{c}, \\ &\qquad\qquad A\boldsymbol{y} = \boldsymbol{b}, \ \boldsymbol{y} \geq 0. \end{aligned} \tag{15}$$

Introducing a slack variable vector $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)^{\mathrm{T}} \geq 0$ such that $A^{\mathrm{T}}\boldsymbol{u} - \boldsymbol{s} = \boldsymbol{c}$, erasing $\boldsymbol{c}$ and changing the objective function to its deviation from 1, (15) is reduced to

$$\begin{aligned} &\underset{\boldsymbol{y},\,\boldsymbol{u},\,\boldsymbol{s}}{\text{maximize}} \quad \frac{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{x}^0}{\boldsymbol{b}^{\mathrm{T}}\boldsymbol{u}}, \\ &\text{sub. to} \quad DA^{\mathrm{T}}\boldsymbol{u} - D\boldsymbol{s} \leq \boldsymbol{g}, \ A\boldsymbol{y} = \boldsymbol{b}, \\ &\qquad\qquad \boldsymbol{y} \geq 0, \ \boldsymbol{s} \geq 0, \ \boldsymbol{s}^{\mathrm{T}}\boldsymbol{y} = 0. \end{aligned} \tag{16}$$

For ensuring the boundedness of the relaxed problems of (16), we add the following redundant constraint:

$$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{u} \leq (\boldsymbol{c}^{\mathrm{R}})^{\mathrm{T}}\boldsymbol{y}, \tag{17}$$

where $\boldsymbol{c}^{\mathrm{R}} \geq \boldsymbol{c}$, for all $\boldsymbol{c} \in \Gamma$. For example, $\boldsymbol{c}^{\mathrm{R}} = (c_1^{\mathrm{R}}, c_2^{\mathrm{R}}, \ldots, c_n^{\mathrm{R}})^{\mathrm{T}}$ can be obtained by

$$c_i^{\mathrm{R}} = \max_{\boldsymbol{c} \in \Gamma} c_i, \ i = 1, 2, \ldots, n. \tag{18}$$

The redundancy of (17) is guaranteed by the duality theorem of LP (see, for example, [9]) and the non-negativity of $\boldsymbol{y}$. Thus, we consider a problem,

$$\begin{aligned} &\underset{\boldsymbol{y},\,\boldsymbol{u},\,\boldsymbol{s}}{\text{maximize}} \quad \frac{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{x}^0}{\boldsymbol{b}^{\mathrm{T}}\boldsymbol{u}}, \\ &\text{sub. to} \quad DA^{\mathrm{T}}\boldsymbol{u} - D\boldsymbol{s} \leq \boldsymbol{g}, \ A\boldsymbol{y} = \boldsymbol{b}, \\ &\qquad\qquad \boldsymbol{b}^{\mathrm{T}}\boldsymbol{u} \leq (\boldsymbol{c}^{\mathrm{R}})^{\mathrm{T}}\boldsymbol{y}, \\ &\qquad\qquad \boldsymbol{y} \geq 0, \ \boldsymbol{s} \geq 0, \ \boldsymbol{s}^{\mathrm{T}}\boldsymbol{y} = 0. \end{aligned} \tag{19}$$

Consider the following problem:

$$\begin{aligned} &\underset{\boldsymbol{q},\,\boldsymbol{u},\,\boldsymbol{s}}{\text{maximize}} \quad \boldsymbol{w}^{\mathrm{T}}\boldsymbol{x}^0, \\ &\text{sub. to} \quad DA^{\mathrm{T}}\boldsymbol{v} - D\boldsymbol{w} \leq \boldsymbol{g}t, \\ &\qquad\qquad A\boldsymbol{q} = \boldsymbol{b}t, \ \boldsymbol{q} \geq 0, \\ &\qquad\qquad \boldsymbol{b}^{\mathrm{T}}\boldsymbol{v} = 1, \ \boldsymbol{w} \geq 0, \ t \geq 0, \\ &\qquad\qquad (\boldsymbol{c}^{\mathrm{R}})^{\mathrm{T}}\boldsymbol{q} \geq 1, \ \boldsymbol{w}^{\mathrm{T}}\boldsymbol{q} = 0. \end{aligned} \tag{20}$$

We have the following theorem.

Theorem 1 *Problem (20) is bounded. Let $(\boldsymbol{q}^*, \boldsymbol{v}^*, \boldsymbol{w}^*, t^*)$ be the optimal solution to (20). We have $t^* > 0$ and a solution $(\boldsymbol{y}^*, \boldsymbol{u}^*, \boldsymbol{s}^*)$ is an optimal solution to (19), where*

$$\boldsymbol{y}^* = \frac{\boldsymbol{q}^*}{t^*}, \ \boldsymbol{u}^* = \frac{\boldsymbol{v}^*}{t^*}, \ \boldsymbol{s}^* = \frac{\boldsymbol{w}^*}{t^*}. \tag{21}$$

Conversely, let $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{s}})$ be the optimal solution to (19). A solution $(\hat{\boldsymbol{q}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}, \hat{t})$ is an optimal solution to (20), where

$$\hat{t} = \frac{1}{\hat{\boldsymbol{b}}^{\mathrm{T}}\hat{\boldsymbol{u}}}, \ \hat{\boldsymbol{q}} = \hat{\boldsymbol{y}}\hat{t}, \ \hat{\boldsymbol{v}} = \hat{\boldsymbol{u}}\hat{t}, \ \hat{\boldsymbol{w}} = \hat{\boldsymbol{s}}\hat{t}. \tag{22}$$

Theorem 1 means that solving (19) is equivalent with solving (20). Therefore, we discuss a solution algorithm for (19) through solving (20).

From the non-negativities of $\boldsymbol{w}$ and $\boldsymbol{y}$, the last complementary constraint $\boldsymbol{w}^{\mathrm{T}}\boldsymbol{q} = 0$ is equivalent to $w_j q_j = 0, \ j = 1, 2, \ldots, n$. If we erase the last constraint $\boldsymbol{w}^{\mathrm{T}}\boldsymbol{y} = 0$ from (20), the relaxed problem becomes an LP problem. The LP problem is called (RP) in what follows. Thus, applying a branch and bound method, (14) can be solved.

Before describing the concrete algorithm, we show a theorem which enhances the lower bound estimation in the branch and bound procedure.

Theorem 2 *Problem (RP) is bounded. Let $(\bar{q}, \bar{v}, \bar{w}, \bar{t})$ be a feasible solution of (RP). We have $\bar{t} > 0$ and there exists a feasible solution $(\check{q}, \check{v}, \check{w}, \check{t})$ of (20) whose objective function value is not less than $(\bar{w}^{\mathrm{T}} x^0 \bar{t} - \bar{w}^{\mathrm{T}} \bar{q})/(\bar{t} - \bar{w}^{\mathrm{T}} \bar{q})$, i.e.,*

$$\check{w}^{\mathrm{T}} x^0 \geq \frac{\bar{w}^{\mathrm{T}} x^0 \bar{t} - \bar{w}^{\mathrm{T}} \bar{q}}{\bar{t} - \bar{w}^{\mathrm{T}} \bar{q}}. \tag{23}$$

Based on the above discussion, Step 2 of Algorithm 1 can be done by the following algorithm.

[Algorithm 2]

Step 1 Set $\tilde{r} = -\infty$ and $\mathcal{P} = \emptyset$. Let (q^*, v^*, w^*, t^*) be an optimal solution to (RP).

Step 2 If $(w^*)^{\mathrm{T}} q^* = 0$, then terminate with letting

$$Ra(x^0) = 1 - (w^*)^{\mathrm{T}} x^0,$$
$$c^k = \frac{A^{\mathrm{T}} v^* - w^*}{t^*}, \quad \text{and}$$
$$q^k = \frac{q^*}{t^*}.$$

Step 3 Select j such that $w_j^* q_j^* > 0$. Generate an LP problem (P_1) by adding $w_j = 0$ to (RP) and an LP problem (P_2) by adding $q_j = 0$ to (RP). Update $\mathcal{P}$ by $\mathcal{P} = \mathcal{P} \cup \{(\mathrm{P}_1), (\mathrm{P}_2)\}$.

Step 4 If $\mathcal{P} = \emptyset$, then terminate with letting

$$Ra(x^0) = 1 - \tilde{r},$$
$$c^k = \frac{A^{\mathrm{T}} \tilde{v} - \tilde{w}}{\tilde{t}} \quad \text{and}$$
$$y^k = \frac{\tilde{q}}{\tilde{t}}.$$

Step 5 Select a problem (P) from $\mathcal{P}$ and set $\mathcal{P} = \mathcal{P} - \{(\mathrm{P})\}$. By a post-optimization technique, solve (P) and let (q^*, v^*, w^*, t^*) be an optimal solution.

Step 6 If $(w^*)^{\mathrm{T}} x^0 \leq \tilde{r}$, then return to Step 4.

Step 7 If

$$\frac{(w^*)^{\mathrm{T}} x^0 t^* - (w^*)^{\mathrm{T}} q^*}{t^* - (w^*)^{\mathrm{T}} q^*} > \tilde{r},$$

then set

$$\tilde{r} = \frac{(w^*)^{\mathrm{T}} x^0 t^* - (w^*)^{\mathrm{T}} q^*}{t^* - (w^*)^{\mathrm{T}} q^*},$$

$\tilde{q} = q^*$, $\tilde{v} = v^*$ and $\tilde{w} = w^*$, $\tilde{t} = t^*$. Return to Step 4. Here,

$$\frac{(w^*)^{\mathrm{T}} x^0 t^* - (w^*)^{\mathrm{T}} q^*}{t^* - (w^*)^{\mathrm{T}} q^*}$$

is a lower bound value of $1 - Ra(x^0)$.

Step 8 Select j such that $w_j^* q_j^* > 0$. Generate an LP problem (P_1) by adding a constraint $w_j = 0$ to (P) and an LP problem (P_2) by adding a constraint $q_j = 0$ to (P). Update $\mathcal{P}$ by $\mathcal{P} = \mathcal{P} \cup \{(\mathrm{P}_1), (\mathrm{P}_2)\}$ and return to Step 4.

4. A numerical example

In order to explain the procedure of the proposed solution algorithm, let us solve the LP problem with convex polyhedral objective coefficients (5) in Example 1. The problem can be rewritten as

$$\begin{aligned}
\text{maximize} \quad & \gamma_1 x_1 + \gamma_2 x_2 \\
& \quad + \gamma_3 x_3 + \gamma_4 x_4 + \gamma_5 x_5, \\
\text{sub. to} \quad & x_1 + x_2 + x_3 = 12, \\
& 3x_1 + x_2 + x_4 = 24, \\
& x_2 + x_5 = 9, \\
& x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0.
\end{aligned} \tag{24}$$

The associated Γ can be written as

$$\Gamma = \{(c_1, c_2, 0, 0, 0)^{\mathrm{T}} \mid 7c_1 - 5c_2 \leq 4,\ c_2 \leq 2,\ -3c_1 + 5c_2 \geq 2,\ c_1 \geq 1\}. \tag{25}$$

By a simple computation, we obtain $c^{\mathrm{R}} = (2, 2, 0, 0, 0)^{\mathrm{T}}$. Solving a linear programming problem,

$$\begin{aligned}
\text{maximize} \quad & 2x_1 + 2x_2 \\
\text{sub. to} \quad & x_1 + x_2 + x_3 = 12, \\
& 3x_1 + x_2 + x_4 = 24, \\
& x_2 + x_5 = 9, \\
& x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0,
\end{aligned} \tag{26}$$

we obtain a feasible solution $x = (6, 6, 0, 0, 3)^{\mathrm{T}}$. This feasible solution is used as the initial solution of Algorithm 1.

We applied Algorithm 1 with $\varepsilon = 0.00001$. The computation process of the maximin achievement rate solution is shown in Fig. 2. The bilevel programming problems at Step 2 were solved by Algorithm 2. As shown in Fig. ??, a maximin achievement rate solution is obtained after introductions of $(c_1, c_2) = (1, 2)$ and $(c_1, c_2) = (1.5, 1.3)$ which are two critical extreme points of Γ as shown in Fig. 1. The obtained maximin achievement rate solution is

$$(x_1, x_2, x_3, x_4, x_5)^{\mathrm{T}} = (3.6, 8.4, 0, 4.8, 0.6)^{\mathrm{T}}, \tag{27}$$

and shown in Fig. 1. As shown in Fig. 1, reflecting the shape of Γ, i.e., lack of the right lower part, the obtained solution is located near an extreme point $(x_1, x_2) = (3, 9)$ rather than $(x_1, x_2) = (6, 6)$.

5. Conclusion

An LP problem with convex polyhedral objective coefficients has been treated. A significance of a maximin achievement rate solution has been

Step 1 Set initial solution as $\boldsymbol{x}^0 = (6,6,0,0,3)^{\mathrm{T}}$. Set $k = 1$ and $r^0 = 1$.

Step 2 Solve a bilevel programming problem (21 LP problems are examined). Set $Ra(\boldsymbol{x}^0) = 0.857143$, $\boldsymbol{c}^1 = (1,2,0,0,0)^{\mathrm{T}}$ and $\boldsymbol{y}^1 = (3,9,0,6,0)^{\mathrm{T}}$.

Step 3 Since $Ra(\boldsymbol{x}^0) = 0.857143 < r^0 = 1 - 0.00001$, continue.

Step 4 Solve an LP problem with 4 constraints. The solution is $\boldsymbol{x}^0 = (3,9,0,6,0)^{\mathrm{T}}$, and $r^0 = 1$. Reset $k = k + 1 = 2$ and return to Step 2.

Step 2 Solve a bilevel programming problem (19 LP problems are examined). Set $Ra(\boldsymbol{x}^0) = 0.964286$, $\boldsymbol{c}^2 = (1.5,1.3,0,0,0)^{\mathrm{T}}$ and $\boldsymbol{y}^2 = (6,6,0,0,3)^{\mathrm{T}}$.

Step 3 Since $Ra(\boldsymbol{x}^0) = 0.964286 < r^0 = 1 - 0.00001$, continue.

Step 4 Solve an LP problem with 5 constraints. The solution is $\boldsymbol{x}^0 = (3.6,8.4,0,4.8,0.6)^{\mathrm{T}}$ and $r^0 = 0.971429$. Reset $k = k + 1 = 3$ and return to Step 2.

Step 2 Solve a bilevel programming problem (19 LP problems are examined). Set $Ra(\boldsymbol{x}^0) = 0.971429$, $\boldsymbol{c}^3 = (1.5,1.3,0,0,0)^{\mathrm{T}}$ and $\boldsymbol{y}^3 = (6,6,0,0,3)^{\mathrm{T}}$.

Step 3 Since $Ra(\boldsymbol{x}^0) = 0.971429 \geq r^0 = 0.971429 - 0.00001$, terminate. The obtained maximin achievement rate solution is

$$(x_1,x_2,x_3,x_4,x_5)^{\mathrm{T}} = (3.6,8.4,0,4.8,0.6)^{\mathrm{T}}.$$

Fig. 2 Computation process

reviewed and a computation method of a maximin achievement rate solution has been proposed. The proposed method is based on a relaxation procedure and a branch and bound method. It has been shown that a maximin achievement rate solution can be obtained by the repetitional use of simplex method.

Finally, the achievement approach can be introduced to an LP problem with a convex polyhedral conic fuzzy vector. A maximin achievement rate solution can also been defined even in this case. The computation method of the solution can be designed based on the proposed algorithms. This would be a future topic of our research.

References

1. M. Inuiguchi, M. Sakawa. "Minimax regret solutions to linear programming problems with an interval objective function", European Journal of Operational Research, **86**, 526–536, 1995.
2. M. Inuiguchi and M. Sakawa. "An achievement rate approach to linear programming problems with an interval objective function", Journal of Operational Research Society, **48**, 25–33, 1997.
3. H. Rommelfanger, R. Hanuscheck, J. Wolf. "Linear programming with fuzzy objectives", Fuzzy Sets and Systems, **29**, 31–48, 1989.
4. N. Furukawa. "Parametric orders on fuzzy numbers and their roles in fuzzy optimization problems", Optimization, **40**, 171–192, 1997.
5. M. Inuiguchi and M. Sakawa. "Possible and necessary optimality tests in possibilistic linear programming problems", Fuzzy Sets and Systems, **67**, 29–46, 1994.
6. K. Shimizu, E. Aiyoshi. "Necessary conditions for min-max problems and algorithms by a relaxation procedure", IEEE Trans. Automatic Control, AC-**25** 62–66, 1980.
7. K. Shimizu, E. Aiyoshi. "A new computational method for Stackelberg and min-max problems", IEEE Trans. Automatic Control, AC-**26** 460–466, 1981.
8. O. L. Mangasarian, Nonlinear Programming, McGraw-Hill, New York, 1969.
9. D. Goldfarb, M. J. Todd. "Linear programming", in: G. L. Nemhauser et al. (eds.), Handbooks in Operations Research and Management Science: Vol.1 Optimization, North-Holland, Amsterdam, 73–170, 1989.