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# SOME GENERALIZATIONS OF SUGENO'S FUZZY INTEGRAL TO SET-VALUED MAPPINGS

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**ABSTRACT.** In this paper we introduce the concept of fuzzy integrals for set-valued mappings, which is an extension of fuzzy integrals for single-valued functions defined by Sugeno. And we give some properties including convergence theorems on fuzzy integrals for set-valued mappings.

*Keywords:* Fuzzy integral, Measurable selection, Fuzzy measure, Set-valued mapping, Fatou's Lemma, Lebesgue convergence theorem.

## 1. Introduction

Since Aumann [1] introduced integrals for set-valued mappings, several kinds of integrals for set-valued mappings have been studied by many authors [4,5,6,7]. In fact, they are all based on the classical Lebesgue integral.

In 1974 Sugeno [9] introduced the concepts of fuzzy measures and fuzzy integrals for single-valued mappings, which are useful in several applied fields like mathematical

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX economics, optimal control theory and engineering. In particular, they have been studied by Ralescu and Adams [8], Wang [10], and others.

On the other hand, using the approaches of Aumann, Zhang and Wang [14] extended fuzzy integrals of Sugeno to set-valued mappings and considered many properties.

In this paper, we extend fuzzy integrals of Sugeno to set-valued mappings, which are different from those by Zhang and Wang [14]. And we discuss properties of our integrals.

In the sequel we will use the following concepts and notations. Let $(\Omega, \Sigma, m)$ is a classical complete and (nonfuzzy) finite measure space. Let $\mu : \Sigma \longrightarrow [0,1](= I)$ be a fuzzy measure in the sense of Sugeno [9], i.e., it satisfies the following conditions:

(i) $\mu(\emptyset) = 0$

(ii) $\mu(A) \leq \mu(B)$ whenever $A \subset B$, $A, B \in \Sigma$

(iii) If $A_1 \subset A_2 \subset \cdots \subset A_n \cdots, A_n \in \Sigma$, then $\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n \longrightarrow \infty} \mu(A_n)$

(iv) If $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots, A_n \in \Sigma$, then $\lim_{n \longrightarrow \infty} \mu(A_n) = \mu(\cap_{n=1}^{\infty} A_n)$.

In addition, we assume that $\mu$ satisfies the following two conditions; for $A, B \in \Sigma$

(i) $\mu$ is null-additive, i.e., $\mu(A) = 0$ implies $\mu(A \cup B) = \mu(B)$,

(ii) $\mu \ll m$, i.e., $m(A) = 0$ implies $\mu(A) = 0$.

A set-valued mapping $F$ is a mapping from $\Omega$ to $2^I \setminus \{\emptyset\}$, and it is measurable if its graph is measurable, i.e.,

$$\{(\omega, r) \in \Omega \times I : r \in F(\omega)\} \in \Sigma \times \mathcal{B},$$

where $\mathcal{B}$ is the Borel algebra of $I$.

$S(F) = \{f | f(\omega) \in F(\omega)\ m - a.e.\ \omega \in \Omega \textit{ and } f : \textit{measurable}\}$ is the family of $m$-a.e. measurable selections of $F$. It is known that $S(F)$ is a closed subset of $I^{\Omega}$.

## 2. Fuzzy integrals for set-valued mappings

In this section we give the definition of fuzzy integrals for set-valued mappings and investigate their properties.

**Definition 2.1.** *Let $F : \Omega \longrightarrow 2^I \setminus \{\emptyset\}$ be a measurable set-valued mapping and $A \in \Sigma$. Then the fuzzy integral of $F$ on $A$ is defined as*

$$\int_A F d\mu = \vee_{\alpha \in I}[\alpha \wedge \mu(A \cap F_\alpha)],$$

*where $F_\alpha = \{\omega \in \Omega : F(\omega) \cap [\alpha, 1] \neq \emptyset\}$.*

*Remark A.* Definition 2.1 is a generalization of the following Sugeno's definition to set-valued mapping:

The fuzzy integral of a measurable single-valued function $f : \Omega \longrightarrow I$ on $A \in \Sigma$ is defined as

$$\int_A f d\mu = \vee_{\alpha \in I}[\alpha \wedge \mu(A \cap f_\alpha)],$$

where $f_\alpha = \{\omega \in \Omega : f(\omega) \geq \alpha\}$.

**Proposition 2.2.** *$\int_A F d\mu = \int_\Omega \chi_A \cdot F d\mu$, where*

$$(\chi_A \cdot F)(\omega) = \begin{cases} F(\omega), \text{ if } \omega \in A \\ \{0\}, \text{ if } \omega \notin A. \end{cases}$$

**Proposition 2.3.** *Let $F$ be a measurable set-valued mapping. If $\mu(A) = 0$, then $\int_A F d\mu = 0$.*

By Proposition 2.2, sometimes we only discuss the integral on $\Omega$. And instead of $\int_\Omega F d\mu$, we will write $\int F d\mu$.

**Definition 2.4.** *Let $F$ and $G$ be measurable set-valued mappings. If $F(\omega) = G(\omega)$ for $\omega \in \Omega$, $m$-a.e., then we say that $F$ equals $G$ $m$-a.e., simply write by $F = G$ $m$-a.e..*

**Lemma 2.5.** *Let $F$ and $G$ be measurable set-valued mappings such that $F = G$ $m$-a.e.. Then $\mu(F_\alpha) = \mu(G_\alpha)$.*

From Lemma 2.5 we can obtain the following theorem.

**Theorem 2.6.** *Let $F$ and $G$ be measurable set-valued mappings. If $F = G$ $m$-a.e., then $\int Fd\mu = \int Gd\mu$.*

**Theorem 2.7.** *Let $F : \Omega \longrightarrow 2^I \setminus \{\emptyset\}$ be a measurable set-valued mapping with closed values. Then the following hold:*

*(i) $\int Fd\mu = \beta \wedge \mu(F_\beta)$ for some $\beta \in I$.*

*(ii) $(\sup_{f\in S(F)} f)(\omega) \geq \beta$ for all $\omega \in F_\beta$.*

**Theorem 2.8.** *Let $F : \Omega \longrightarrow 2^I \setminus \{\emptyset\}$ be a measurable set-valued mapping with closed values. Then*

$$\int Fd\mu = \int \sup_{f\in S(F)} fd\mu.$$

From the above theorem, we obtain the following two propositions.

**Propsoition 2.9.** *Let $F : \Omega \longrightarrow 2^I \setminus \{\emptyset\}$ be a measurable set-valued mapping with closed values and $c \in I$. Then*

$$\int (c \vee F)d\mu = \int c\ d\mu \vee \int Fd\mu,$$

*where $(c \vee F)(\omega) = \{c \vee f(\omega) | f(\omega) \in F(\omega)\}$ for $\omega \in \Omega$.*

**Proposition 2.10.** *Let $F_1, F_2 : \Omega \longrightarrow 2^I \setminus \{\emptyset\}$ be measurable set-valued mappings with closed values. If $F_1 \subset F_2$ (i.e., $F_1(\omega) \subset F_2(\omega)$ for each $\omega \in \Omega$), then $\int F_1 d\mu \leq \int F_2 d\mu$.*

**Corollary 2.11.** *Let $F : \Omega \longrightarrow 2^I \setminus \{\emptyset\}$ be a measurable set-valued mapping with closed values. Then there exists $g \in S(F)$ such that $\int Fd\mu = \int gd\mu$.*

**Example 2.12.** *Define a set-valued mapping* $F : [0,2] \longrightarrow 2^I$ *by*

$$F(\omega) = \begin{cases} \frac{3}{4} & \text{if } \omega \in [0, \frac{1}{2}) \cup (1,2] \\ [\frac{1}{2}, 1] & \text{if } \omega = \frac{1}{2}, 1 \\ \{\frac{1}{2}, 1\} & \text{otherwise.} \end{cases}$$

*Then* $S(F)$ *is compact and* $\int F d\mu = \frac{3}{4} = \int f d\mu$, *where* $f : [0,2] \longrightarrow [0,1]$ *is a function such that*

$$f(\omega) = \begin{cases} \frac{3}{4} & \text{if } \omega \in [0, \frac{1}{2}) \cup (1,2] \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Now we obtain the following existence theorem for solutions to integral equations as an application of Theorem 2.7.

**Theorem 2.13.** *For* $t \in (0,1]$, *a set-valued mapping* $F : \Omega \longrightarrow 2^I \setminus \{\emptyset\}$ *with closed values satisfies the equation*

$$\int F d\mu = t$$

*if and only if* $\beta \wedge \mu(F_\beta) \leq t$ *for all* $0 < \beta \leq 1$ *and there exists* $\beta_0 \in (0,1]$ *such that* $\beta_0 \wedge \mu(F_{\beta_0}) = t$.

**Corollary 2.14 [11].** *For* $t \in (0,1]$, *a measurable function* $f : \Omega \longrightarrow I$ *satisfies the equation*

$$\int f d\mu = t$$

*if and only if* $\beta \wedge \mu(f_\beta) \leq t$ *for all* $0 < \beta \leq 1$ *and there exists* $\beta_0 \in (0,1]$ *such that* $\beta_0 \wedge \mu(f_{\beta_0}) = t$.

Recall that for any sequence $\{A_n\} \subset 2^I$, $\lim_n \sup A_n = \{\alpha \in I : \alpha = \lim_{k \longrightarrow \infty} \alpha_{n_k}, \alpha_n \in A_n\}$ and $\lim_n \inf A_n = \{\alpha \in I : \alpha = \lim_n \alpha_n, \alpha_n \in A_n\}$ are closed sets [2]. If $\lim_n \sup A_n = \lim_n \inf A_n = A$, then we say that $\{A_n\}$ is convergent to $A$.

Let $\{F_n\}$ be a sequence of set-valued mappings from $\Omega$ to $2^I \setminus \{\emptyset\}$. Then we define $\lim_n \sup F_n$, $\lim_n \inf F_n$ and $\lim F_n$ pointwise as

$(\lim_n \sup F_n)(\omega) = \lim_n \sup F_n(\omega)$, $(\lim_n \inf F_n)(\omega) = \lim_n \inf F_n(\omega)$, $(\lim_n F_n)(\omega) = \lim_n F_n(\omega)$ for $\omega \in \Omega$.

We may assume that $(\liminf F_n)(\omega) \neq \emptyset$ for $\omega \in \Omega$. Then $\lim_n \sup F_n$, $\lim_n \inf F_n$ and $\lim_n F_n$ are set-valued mappings.

## 3. Convergence theorems

In this section we give a convergence theorem of Lebesgue-type for set-valued mappings.

**Theorem 3.1 (Fatou's Lemma).** *Let $\{F_n\}$ be a sequence of measurable set-valued mappings from $\Omega$ to $2^I \setminus \{\emptyset\}$ with closed values. Then the following hold:*

*(i)* $\lim_n \sup \int F_n d\mu \leq \int \lim_n \sup F_n d\mu$.

*(ii)* $\int \lim_n \inf F_n d\mu \leq \lim_n \inf \int F_n d\mu$.

From Fatou's Lemma, we can obtain the following Lebesgue Convergence Theorem.

**Theorem 3.2.** *Let $\{F_n\}$ be a sequence of measurable set-valued mappings from $\Omega$ to $2^I \setminus \{\emptyset\}$ with closed values and $F$ a measurable set-valued mapping from $\Omega$ to $2^I \setminus \{\emptyset\}$ with closed values. If $\lim F_n = F$, then $\lim \int F_n d\mu = \int F d\mu$.*

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