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FUZZY STRONGLY r-SEMICONTINUOUS MAPS

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Abstract

We introduce the concepts of fuzzy strongly r-semiopen sets and fuzzy strongly r-semicontinuous maps, and investigate some of their basic properties.

Keywords: fuzzy strongly r-semiopen, fuzzy strongly r-semicontinuous

1. Introduction

Chang [2] introduced fuzzy topological spaces and several other authors continued the investigation of such spaces. Some authors [4,7] introduced new definitions of fuzzy topology as a generalization of Chang's fuzzy topology. In this paper, we generalize the concepts of fuzzy strongly semiopen sets and fuzzy strongly semicontinuous maps of Zhong [8]. We introduce the concepts of fuzzy strongly r-semiopen sets and fuzzy strongly r-semicontinuous maps, and then investigate some of their basic properties.

2. Preliminaries

In this paper, I will denote the unit interval $[0,1]$ of the real line and $I_0 = (0,1]$. A member μ of I^X is called a fuzzy set of X. For any $\mu \in I^X$, μ^c denotes the complement $1-\mu$. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on X with value 0 and 1, respectively. All other notations are standard notations of fuzzy set theory.

A *Chang's fuzzy topology* on X is a family T of fuzzy sets in X which satisfies the following properties:

(1) $\tilde{0}, \tilde{1} \in T$.

(2) If $\mu_1, \mu_2 \in T$ then $\mu_1 \wedge \mu_2 \in T$.

(3) If $\mu_i \in T$ for each i, then $\bigvee \mu_i \in T$.

The pair (X,T) is called a *Chang's fuzzy topological space.*

A *fuzzy topology* on X is a map $\mathcal{T} : I^X \to I$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$.

(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$.

(3) $\mathcal{T}(\bigvee \mu_i) \geq \bigwedge \mathcal{T}(\mu_i)$.

The pair $(X,\mathcal{T})$ is called a *fuzzy topological space.*

Definition 2.1([5]) Let μ be a fuzzy set of a fuzzy topological space $(X,\mathcal{T})$ and $r \in I_0$. Then μ is called

(1) a *fuzzy r-open set* of X if $\mathcal{T}(\mu) \geq r$,

(2) a *fuzzy r-closed set* of X if $\mathcal{T}(\mu^c) \geq r$.

Definition 2.2([3]) Let $(X,\mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-closure* is defined by

$$\mathrm{cl}(\mu,r) = \bigwedge\{\rho \in I^X : \mu \leq \rho, \mathcal{T}(\rho^c) \geq r\}.$$

Definition 2.3([5]) Let $(X,\mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-interior* is defined by

$$\mathrm{int}(\mu,r) = \bigvee\{\rho \in I^X : \mu \geq \rho, \mathcal{T}(\rho) \geq r\}.$$

Theorem 2.4([5]) *For a fuzzy set μ of a fuzzy topological space $(X,\mathcal{T})$ and $r \in I_0$, we have:*

(1) $\mathrm{int}(\mu,r)^c = \mathrm{cl}(\mu^c,r)$.

(2) $\mathrm{cl}(\mu,r)^c = \mathrm{int}(\mu^c,r)$.

Definition 2.5([5,6]) Let μ be a fuzzy set of a fuzzy topological space $(X,\mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy r-semiopen* if there is a fuzzy r-open set ρ in X such that $\rho \leq \mu \leq \mathrm{cl}(\rho,r)$,

(2) *fuzzy r-semiclosed* if there is a fuzzy r-closed set ρ in X such that $\mathrm{int}(\rho,r) \leq \mu \leq \rho$,

(3) *fuzzy r-preopen* if $\mu \leq \text{int}(\text{cl}(\mu, r), r)$,

(4) *fuzzy r-preclosed* if $\text{cl}(\text{int}(\mu, r), r) \leq \mu$.

Definition 2.6([5,6]) Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is called

(1) a *fuzzy r-continuous* map if $f^{-1}(\mu)$ is a fuzzy r-open set of X for each fuzzy r-open set μ of Y,

(2) a *fuzzy r-semicontinuous* map if $f^{-1}(\mu)$ is a fuzzy r-semiopen set of X for each fuzzy r-open set μ of Y,

(3) a *fuzzy r-precontinuous* map if $f^{-1}(\mu)$ is a fuzzy r-preopen set of X for each fuzzy r-open set μ of Y.

3. Fuzzy strongly r-semiopen sets

We are going to define fuzzy strongly r-semiopen sets and fuzzy strongly r-semiclosed sets, and then investigate some of their properties.

Definition 3.1 Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy strongly r-semiopen* if there is a fuzzy r-open set ρ in X such that

$$\rho \leq \mu \leq \text{int}(\text{cl}(\rho, r), r),$$

(2) *fuzzy strongly r-semiclosed* if there is a fuzzy r-closed set ρ in X such that

$$\text{cl}(\text{int}(\rho, r), r) \leq \mu \leq \rho.$$

Theorem 3.2 *Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then the following statements are equivalent:*

(1) *μ is fuzzy strongly r-semiopen.*

(2) *μ^c is fuzzy strongly r-semiclosed.*

(3) $\mu \leq \text{int}(\text{cl}(\text{int}(\mu, r), r), r)$.

(4) $\mu^c \geq \text{cl}(\text{int}(\text{cl}(\mu^c, r), r), r)$.

(5) *μ is fuzzy r-semiopen and fuzzy r-preopen.*

(6) *μ^c is fuzzy r-semiclosed and fuzzy r-preclosed.*

Proof. (1) ⇔ (2), (3) ⇔ (4) and (5) ⇔ (6) follows from Theorem 2.4.

(1) ⇒ (3) Let μ be a fuzzy strongly r-semiopen set of X. Then there is a fuzzy r-open set ρ in X such that $\rho \leq \mu \leq \text{int}(\text{cl}(\rho, r), r)$. Since $\mathcal{T}(\rho) \geq r$ and $\mu \geq \rho$, $\rho = \text{int}(\rho, r) \leq \text{int}(\mu, r)$. Thus we have

$$\mu \leq \text{int}(\text{cl}(\rho, r), r) \leq \text{int}(\text{cl}(\text{int}(\mu, r), r), r).$$

(3) ⇒ (1) Let $\text{int}(\text{cl}(\text{int}(\mu, r), r), r) \geq \mu$ and take $\rho = \text{int}(\mu, r)$. Since $\mathcal{T}(\text{int}(\mu, r)) \geq r$, ρ is a fuzzy r-open set. Also,

$$\begin{aligned} \rho &= \text{int}(\mu, r) \leq \mu \\ &\leq \text{int}(\text{cl}(\text{int}(\mu, r), r), r) = \text{int}(\text{cl}(\rho, r), r). \end{aligned}$$

Hence μ is a fuzzy strongly r-semiopen set.

(1) ⇒ (5) It is obvious.

(5) ⇒ (3) Let μ be fuzzy r-semiopen and fuzzy r-preopen. Then $\mu \leq \text{cl}(\text{int}(\mu, r), r)$ and $\mu \leq \text{int}(\text{cl}(\mu, r), r)$. Thus

$$\begin{aligned} \mu &\leq \text{int}(\text{cl}(\mu, r), r) \\ &\leq \text{int}(\text{cl}(\text{cl}(\text{int}(\mu, r), r), r), r) \\ &= \text{int}(\text{cl}(\text{int}(\mu, r), r), r). \end{aligned}$$

Theorem 3.3 (1) *Any union of fuzzy strongly r-semiopen sets is fuzzy strongly r-semiopen.*
(2) *Any intersection of fuzzy strongly r-semiclosed sets is fuzzy strongly r-semiclosed.*

Proof. (1) Let $\{\mu_i\}$ be a collection of fuzzy strongly r-semiopen sets. Then for each i, there is a fuzzy r-open set ρ_i such that $\rho_i \leq \mu_i \leq \text{int}(\text{cl}(\rho_i, r), r)$. Since $\mathcal{T}(\bigvee \rho_i) \geq \bigwedge \mathcal{T}(\rho_i) \geq r$, $\bigvee \rho_i$ is a fuzzy r-open set. Also

$$\begin{aligned} \bigvee \rho_i &\leq \bigvee \mu_i \\ &\leq \bigvee \text{int}(\text{cl}(\rho_i, r), r) \\ &\leq \text{int}(\text{cl}(\bigvee \rho_i, r), r). \end{aligned}$$

Hence $\bigvee \mu_i$ is a fuzzy strongly r-semiopen set.

(2) It follows from (1) using Theorem 3.2.

Remark 3.4 It is obvious that every fuzzy r-open set is fuzzy strongly r-semiopen and every fuzzy strongly r-semiopen set is not only a fuzzy r-semiopen set but also a fuzzy r-preopen set. That all of the converses need not be true is shown by the following example.

Example 3.5 Let $X = I$ and μ_1, μ_2 and μ_3 be fuzzy sets of X defined by

$$\mu_1(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1; \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{4}, \\ -4x + 2 & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1; \end{cases}$$

and

$$\mu_3(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{4}, \\ \frac{1}{3}(4x-1) & \text{if } \frac{1}{4} \le x \le 1. \end{cases}$$

Define $\mathcal{T}_1 : I^X \to I$, $\mathcal{T}_2 : I^X \to I$ and $\mathcal{T}_3 : I^X \to I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathcal{T}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \mu_2, \mu_1 \vee \mu_2, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\mathcal{T}_3(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_3, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are fuzzy topologies on X. The fuzzy set $\mu_3(\mu_3^c)$ is fuzzy strongly $\frac{1}{2}$-semiopen ($\frac{1}{2}$-semiclosed) which is not fuzzy $\frac{1}{2}$-open ($\frac{1}{2}$-closed) in $(X, \mathcal{T}_1)$. Also $\mu_3(\mu_3^c)$ is fuzzy $\frac{1}{2}$-semiopen ($\frac{1}{2}$-semiclosed) but not fuzzy strongly $\frac{1}{2}$-semiopen ($\frac{1}{2}$-semiclosed) in $(X, \mathcal{T}_2)$. It can be also seen that $\mu_1(\mu_1^c)$ is fuzzy $\frac{1}{2}$-preopen ($\frac{1}{2}$-preclosed) which is not fuzzy strongly $\frac{1}{2}$-semiopen ($\frac{1}{2}$-semiclosed) in $(X, \mathcal{T}_3)$. The example further shows that μ_1 in $(X, \mathcal{T}_3)$ is a fuzzy $\frac{1}{2}$-preopen set which is not fuzzy $\frac{1}{2}$-semiopen and μ_3 in $(X, \mathcal{T}_2)$ is a fuzzy $\frac{1}{2}$-semiopen set which is not fuzzy $\frac{1}{2}$-preopen.

Definition 3.6 Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy strong r-semiclosure* is defined by

$$\text{sscl}(\mu, r) = \bigwedge\{\rho \in I^X : \mu \le \rho,\ \rho \text{ is fuzzy strongly } r\text{-semiclosed}\}$$

and the *fuzzy strong r-semiinterior* is defined by

$$\text{ssint}(\mu, r) = \bigvee\{\rho \in I^X : \mu \ge \rho,\ \rho \text{ is fuzzy strongly } r\text{-semiopen}\}.$$

Obviously $\text{sscl}(\mu, r)$ is the smallest fuzzy strongly r-semiclosed set which contains μ and $\text{ssint}(\mu, r)$ is the greatest fuzzy strongly r-semiopen set which is contained in μ. Moreover, $\text{sscl}(\mu, r) = \mu$ for any fuzzy strongly r-semiclosed set μ and $\text{ssint}(\mu, r) = \mu$ for any fuzzy strongly r-semiopen set μ. Also we have

$$\text{int}(\mu, r) \le \text{ssint}(\mu, r) \le \mu \le \text{sscl}(\mu, r) \le \text{cl}(\mu, r).$$

Moreover, we have the following results.

(1) $\text{ssint}(\tilde{0}, r) = \tilde{0}, \text{ssint}(\tilde{1}, r) = \tilde{1}$.

(2) $\text{ssint}(\mu, r) \le \mu$.

(3) $\text{ssint}(\mu \wedge \rho, r) \le \text{ssint}(\mu, r) \wedge \text{ssint}(\rho, r)$.

(4) $\text{ssint}(\text{ssint}(\mu, r), r) = \text{ssint}(\mu, r)$.

(5) $\text{sscl}(\tilde{0}, r) = \tilde{0}, \text{sscl}(\tilde{1}, r) = \tilde{1}$.

(6) $\text{sscl}(\mu, r) \ge \mu$.

(7) $\text{sscl}(\mu \vee \rho, r) \ge \text{sscl}(\mu, r) \vee \text{sscl}(\rho, r)$.

(8) $\text{sscl}(\text{sscl}(\mu, r), r) = \text{sscl}(\mu, r)$.

Theorem 3.7 *For a fuzzy set μ of a fuzzy topological space X and $r \in I_0$,*

(1) $\text{ssint}(\mu, r)^c = \text{sscl}(\mu^c, r)$.

(2) $\text{sscl}(\mu, r)^c = \text{ssint}(\mu^c, r)$.

Proof. (1) Since $\text{ssint}(\mu, r) \le \mu$ and $\text{ssint}(\mu, r)$ is fuzzy strongly r-semiopen, $\mu^c \le \text{ssint}(\mu, r)^c$ and $\text{ssint}(\mu, r)^c$ is fuzzy strongly r-semiclosed in X. Thus $\text{sscl}(\mu^c, r) \le \text{ssint}(\mu, r)^c$. Conversely, since $\mu^c \le \text{sscl}(\mu^c, r)$ and $\text{sscl}(\mu^c, r)$ is fuzzy strongly r-semiclosed in X, $\text{sscl}(\mu^c, r)^c \le \mu$ and $\text{sscl}(\mu^c, r)^c$ is fuzzy strongly r-semiopen in X. Thus $\text{sscl}(\mu^c, r)^c \le \text{ssint}(\mu, r)$ and hence $\text{ssint}(\mu, r)^c \le \text{sscl}(\mu^c, r)$.

(2) Similar to (1).

Theorem 3.8 *Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is fuzzy strongly r-semiopen (r-semiclosed) in $(X, \mathcal{T})$ if and only if μ is fuzzy strongly semiopen (semiclosed) in $(X, \mathcal{T}_r)$.*

Proof. Straightforward.

Theorem 3.9 *Let μ be a fuzzy set of a Chang's fuzzy topological space (X, T) and $r \in I_0$. Then μ is fuzzy strongly semiopen (semiclosed) in (X, T) if and only if μ is fuzzy strongly r-semiopen (semiclosed) in (X, T^r).*

Proof. Straightforward.

4. Fuzzy strongly r-semicontinuous maps

We introduce the notions of fuzzy strongly r-semicontinuous maps, fuzzy strongly r-semiopen maps and fuzzy strongly r-semiclosed maps, and then investigate some of their properties.

Definition 4.1 Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is said to be

(1) *fuzzy strongly r-semicontinuous* if $f^{-1}(\mu)$ is a fuzzy strongly r-semiopen set of X for each fuzzy r-open set μ of Y, or equivalently, $f^{-1}(\mu)$ is a fuzzy strongly r-semiclosed set of X for each fuzzy r-closed set μ of Y,

(2) *fuzzy strongly r-semiopen* if $f(\rho)$ is a fuzzy strongly r-semiopen set of Y for each fuzzy r-open set ρ of X,

(3) *fuzzy strongly r-semiclosed* if $f(\rho)$ is a fuzzy strongly r-semiclosed set of Y for each fuzzy r-closed set ρ of X.

Remark 4.2 Clearly, every fuzzy r-continuous map is also a fuzzy strongly r-semicontinuous map and every fuzzy strongly r-semicontinuous map is not only a fuzzy r-semicontinuous map but also a fuzzy r-precontinuous map. That all of the converses need not be true is shown by the following example.

Example 4.3 Let $X = \{x\}$ and μ_1, μ_2 and μ_3 be fuzzy sets of X defined by

$$\mu_1(x) = \tfrac{1}{2}, \quad \mu_2(x) = \tfrac{1}{3}, \quad \mu_3(x) = \tfrac{1}{4}.$$

Define $\mathcal{T}_1 : I^X \to I$, $\mathcal{T}_2 : I^X \to I$ and $\mathcal{T}_3 : I^X \to I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \mu_3, \\ 0 & \text{otherwise}; \end{cases}$$

$$\mathcal{T}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_2, \\ 0 & \text{otherwise}; \end{cases}$$

and

$$\mathcal{T}_3(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_3, \\ 0 & \text{otherwise}. \end{cases}$$

Then clearly $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are fuzzy topologies on X.

(1) Consider the map $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ defined by $f(x) = x$. Then $f^{-1}(\tilde{0}) = \tilde{0}$, $f^{-1}(\tilde{1}) = \tilde{1}$ and $f^{-1}(\mu_2) = \mu_2$ are fuzzy strongly $\frac{1}{2}$-semiopen sets of $(X, \mathcal{T}_1)$ and hence f is fuzzy strongly $\frac{1}{2}$-semicontinuous. On the other hand, $f^{-1}(\mu_2) = \mu_2$ is not fuzzy $\frac{1}{2}$-open in $(X, \mathcal{T}_1)$ and hence f is not fuzzy $\frac{1}{2}$-continuous.

(2) Consider the map $f : (X, \mathcal{T}_2) \to (X, \mathcal{T}_3)$ defined by $f(x) = x$. Then $f^{-1}(\tilde{0}) = \tilde{0}$, $f^{-1}(\tilde{1}) = \tilde{1}$ and $f^{-1}(\mu_3) = \mu_3$ are fuzzy $\frac{1}{2}$-preopen sets of $(X, \mathcal{T}_2)$ and hence f is fuzzy $\frac{1}{2}$-precontinuous. On the other hand, $f^{-1}(\mu_3) = \mu_3$ is not fuzzy strongly $\frac{1}{2}$-semiopen in $(X, \mathcal{T}_2)$ and hence f is not fuzzy strongly $\frac{1}{2}$-semicontinuous.

(3) Consider the map $f : (X, \mathcal{T}_3) \to (X, \mathcal{T}_2)$ defined by $f(x) = x$. Then $f^{-1}(\tilde{0}) = \tilde{0}$, $f^{-1}(\tilde{1}) = \tilde{1}$ and $f^{-1}(\mu_2) = \mu_2$ are fuzzy $\frac{1}{2}$-semiopen sets of $(X, \mathcal{T}_3)$ and hence f is fuzzy $\frac{1}{2}$-semicontinuous. On the other hand, $f^{-1}(\mu_2) = \mu_2$ is not fuzzy strongly $\frac{1}{2}$-semiopen in $(X, \mathcal{T}_3)$ and hence f is not fuzzy strongly $\frac{1}{2}$-semicontinuous.

The next theorem provides alternative characterizations of a fuzzy strongly r-semicontinuous map by fuzzy r-closure and fuzzy r-interior.

Theorem 4.4 *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy strongly r-semicontinuous map.*

(2) $\text{cl}(\text{int}(\text{cl}(f^{-1}(\mu), r), r), r) \leq f^{-1}(\text{cl}(\mu, r))$ *for each fuzzy set μ of Y.*

(3) $f(\text{cl}(\text{int}(\text{cl}(\rho, r), r), r)) \leq \text{cl}(f(\rho), r)$ *for each fuzzy set ρ of X.*

Proof. (1) $\Rightarrow$ (2) Let f be a fuzzy strongly r-semicontinuous map and μ a fuzzy set of Y. Then $\text{cl}(\mu, r)$ is a fuzzy r-closed set of Y. Since f is fuzzy strongly r-semicontinuous, $f^{-1}(\text{cl}(\mu, r))$ is a fuzzy strongly r-semiclosed set of X. By Theorem 3.2,

$$\begin{aligned} f^{-1}(\text{cl}(\mu, r)) &\geq \text{cl}(\text{int}(\text{cl}(f^{-1}(\text{cl}(\mu, r)), r), r), r) \\ &\geq \text{cl}(\text{int}(\text{cl}(f^{-1}(\mu), r), r), r). \end{aligned}$$

(2) $\Rightarrow$ (3) Let ρ be a fuzzy set of X. Then $f(\rho)$ is a fuzzy set of Y. By (2),

$$\begin{aligned} f^{-1}(\text{cl}(f(\rho), r)) &\geq \text{cl}(\text{int}(\text{cl}(f^{-1}f(\rho), r), r), r) \\ &\geq \text{cl}(\text{int}(\text{cl}(\rho, r), r), r). \end{aligned}$$

Hence we have

$$\begin{aligned} \text{cl}(f(\rho), r) &\geq ff^{-1}(\text{cl}(f(\rho), r)) \\ &\geq f(\text{cl}(\text{int}(\text{cl}(\rho, r), r), r)). \end{aligned}$$

(3) $\Rightarrow$ (1) Let μ be a fuzzy r-closed set of Y. Then $f^{-1}(\mu)$ is a fuzzy set of X. By (3),

$$\begin{aligned} f(\text{cl}(\text{int}(\text{cl}(f^{-1}(\mu), r), r), r)) &\leq \text{cl}(ff^{-1}(\mu), r) \\ &\leq \text{cl}(\mu, r) = \mu. \end{aligned}$$

Hence we have

$$\begin{aligned} &\text{cl}(\text{int}(\text{cl}(f^{-1}(\mu), r), r), r) \\ &\quad \leq f^{-1}f(\text{cl}(\text{int}(\text{cl}(f^{-1}(\mu), r), r), r)) \\ &\quad \leq f^{-1}(\mu). \end{aligned}$$

Thus $f^{-1}(\mu)$ is a fuzzy strongly r-semiclosed set of X and hence f is a fuzzy strongly r-semicontinuous map.

A fuzzy strongly r-semicontinuous map can be characterized as follows.

Theorem 4.5 *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy strongly r-semicontinuous map.*

(2) $f(\text{sscl}(\rho, r)) \leq \text{cl}(f(\rho), r)$ *for each fuzzy set ρ of X.*

(3) $\mathrm{sscl}(f^{-1}(\mu),r) \leq f^{-1}(\mathrm{cl}(\mu,r))$ *for each fuzzy set* μ *of* Y.

(4) $f^{-1}(\mathrm{int}(\mu,r)) \leq \mathrm{ssint}(f^{-1}(\mu),r)$ *for each fuzzy set* μ *of* Y.

Proof. (1) $\Rightarrow$ (2) Let ρ be a fuzzy set of X. Since $\mathrm{cl}(f(\rho),r)$ is a fuzzy r-closed set of Y, $f^{-1}(\mathrm{cl}(f(\rho),r))$ is a fuzzy strongly r-semiclosed set of X. Thus

$$\begin{aligned}\mathrm{sscl}(\rho,r) &\leq \mathrm{sscl}(f^{-1}f(\rho),r)\\ &\leq \mathrm{sscl}(f^{-1}(\mathrm{cl}(f(\rho),r)),r)\\ &= f^{-1}(\mathrm{cl}(f(\rho),r)).\end{aligned}$$

Hence

$$f(\mathrm{sscl}(\rho,r)) \leq ff^{-1}(\mathrm{cl}(f(\rho),r)) \leq \mathrm{cl}(f(\rho),r).$$

(2) $\Rightarrow$ (3) Let μ be a fuzzy set of Y. By (2),

$$f(\mathrm{sscl}(f^{-1}(\mu),r)) \leq \mathrm{cl}(ff^{-1}(\mu),r) \leq \mathrm{cl}(\mu,r).$$

Thus

$$\begin{aligned}\mathrm{sscl}(f^{-1}(\mu),r) &\leq f^{-1}f(\mathrm{sscl}(f^{-1}(\mu),r))\\ &\leq f^{-1}(\mathrm{cl}(\mu,r)).\end{aligned}$$

(3) $\Rightarrow$ (4) Let μ be a fuzzy set of Y. Then μ^c is a fuzzy set of Y. By (3),

$$\mathrm{sscl}(f^{-1}(\mu)^c,r) = \mathrm{sscl}(f^{-1}(\mu^c),r) \leq f^{-1}(\mathrm{cl}(\mu^c,r)).$$

By Theorem 2.4 and Theorem 3.7,

$$\begin{aligned}f^{-1}(\mathrm{int}(\mu,r)) &= f^{-1}(\mathrm{cl}(\mu^c,r))^c\\ &\leq \mathrm{sscl}(f^{-1}(\mu)^c,r)^c\\ &= \mathrm{ssint}(f^{-1}(\mu),r).\end{aligned}$$

(4) $\Rightarrow$ (1) Let μ be a fuzzy r-open set of Y. Then $\mathrm{int}(\mu,r)=\mu$. By (4),

$$\begin{aligned}f^{-1}(\mu) &= f^{-1}(\mathrm{int}(\mu,r))\\ &\leq \mathrm{ssint}(f^{-1}(\mu),r)\\ &\leq f^{-1}(\mu).\end{aligned}$$

So $f^{-1}(\mu) = \mathrm{ssint}(f^{-1}(\mu),r)$ and hence $f^{-1}(\mu)$ is a fuzzy strongly r-semiopen set of X. Thus f is fuzzy strongly r-semicontinuous.

Theorem 4.6 *Let* $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ *be a bijection and* $r \in I_0$. *Then* f *is a fuzzy strongly* r-*semicontinuous map if and only if* $\mathrm{int}(f(\rho),r) \leq f(\mathrm{ssint}(\rho,r))$ *for each fuzzy set* ρ *of* X.

Proof. Let f be a fuzzy strongly r-semicontinuous map and ρ a fuzzy set of X. Since $\mathrm{int}(f(\rho),r)$ is fuzzy r-open in Y, $f^{-1}(\mathrm{int}(f(\rho),r))$ is fuzzy strongly r-semiopen in X. Since f is one-to-one, we have

$$f^{-1}(\mathrm{int}(f(\rho),r)) \leq \mathrm{ssint}(f^{-1}f(\rho),r) = \mathrm{ssint}(\rho,r).$$

Since f is onto, we have

$$\mathrm{int}(f(\rho),r) = ff^{-1}(\mathrm{int}(f(\rho),r)) \leq f(\mathrm{ssint}(\rho,r)).$$

Conversely, let μ be a fuzzy r-open set of Y. Then $\mathrm{int}(\mu,r)=\mu$. Since f is onto,

$$\begin{aligned}f(\mathrm{ssint}(f^{-1}(\mu),r)) &\geq \mathrm{int}(ff^{-1}(\mu),r)\\ &= \mathrm{int}(\mu,r) = \mu.\end{aligned}$$

Since f is one-to-one, we have

$$\begin{aligned}f^{-1}(\mu) &\leq f^{-1}f(\mathrm{ssint}(f^{-1}(\mu),r))\\ &= \mathrm{ssint}(f^{-1}(\mu),r)\\ &\leq f^{-1}(\mu).\end{aligned}$$

So $f^{-1}(\mu) = \mathrm{ssint}(f^{-1}(\mu),r)$ and hence $f^{-1}(\mu)$ is a fuzzy strongly r-semiopen set of X. Thus f is fuzzy strongly r-semicontinuous.

The next theorem provides alternative characterizations of a fuzzy strongly r-semiopen map.

Theorem 4.7 *Let* $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ *be a map and* $r \in I_0$. *Then the following statements are equivalent:*

(1) *f is a fuzzy strongly r-semiopen map.*

(2) $f(\mathrm{int}(\rho,r)) \leq \mathrm{ssint}(f(\rho),r)$ *for each fuzzy set* ρ *of* X.

(3) $\mathrm{int}(f^{-1}(\mu),r) \leq f^{-1}(\mathrm{ssint}(\mu,r))$ *for each fuzzy set* μ *of* Y.

Proof. (1) $\Rightarrow$ (2) Let ρ be a fuzzy set of X. Clearly $\mathrm{int}(\rho,r)$ is a fuzzy r-open set of X. Since f is a fuzzy strongly r-semiopen map, $f(\mathrm{int}(\rho,r))$ is a fuzzy strongly r-semiopen set of Y. Thus

$$f(\mathrm{int}(\rho,r)) = \mathrm{ssint}(f(\mathrm{int}(\rho,r)),r) \leq \mathrm{ssint}(f(\rho),r).$$

(2) $\Rightarrow$ (3) Let μ be a fuzzy set of Y. Then $f^{-1}(\mu)$ is a fuzzy set of X. By (2),

$$f(\mathrm{int}(f^{-1}(\mu),r)) \leq \mathrm{ssint}(ff^{-1}(\mu),r) \leq \mathrm{ssint}(\mu,r).$$

Thus we have

$$\begin{aligned}\mathrm{int}(f^{-1}(\mu),r) &\leq f^{-1}f(\mathrm{int}(f^{-1}(\mu),r))\\ &\leq f^{-1}(\mathrm{ssint}(\mu,r)).\end{aligned}$$

(3) $\Rightarrow$ (1) Let ρ be a fuzzy r-open set of X. Then $\mathrm{int}(\rho,r)=\rho$ and $f(\rho)$ is a fuzzy set of Y. By (3),

$$\begin{aligned}\rho = \mathrm{int}(\rho,r) &\leq \mathrm{int}(f^{-1}f(\rho),r)\\ &\leq f^{-1}(\mathrm{ssint}(f(\rho),r)).\end{aligned}$$

Hence we have

$$f(\rho) \leq ff^{-1}(\mathrm{ssint}(f(\rho),r)) \leq \mathrm{ssint}(f(\rho),r) \leq f(\rho).$$

Thus $f(\rho) = \mathrm{ssint}(f(\rho), r)$ and hence $f(\rho)$ is a fuzzy strongly r-semiopen set of Y. Therefore f is fuzzy strongly r-semiopen.

A fuzzy strongly r-semiclosed map can be characterized as follows.

Theorem 4.8 *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy strongly r-semiclosed map.*

(2) $\mathrm{sscl}(f(\rho), r) \le f(\mathrm{cl}(\rho, r))$ *for each fuzzy set ρ of X.*

Proof. (1) $\Rightarrow$ (2) Let ρ be a fuzzy set of X. Clearly $\mathrm{cl}(\rho, r)$ is a fuzzy r-closed set of X. Since f is a fuzzy strongly r-semiclosed map, $f(\mathrm{cl}(\rho, r))$ is a fuzzy strongly r-semiclosed set of Y. Thus

$$\mathrm{sscl}(f(\rho), r) \le \mathrm{sscl}(f(\mathrm{cl}(\rho, r)), r) = f(\mathrm{cl}(\rho, r)).$$

(2) $\Rightarrow$ (1) Let ρ be a fuzzy r-closed set of X. Then $\mathrm{cl}(\rho, r) = \rho$. By (2),

$$\mathrm{sscl}(f(\rho), r) \le f(\mathrm{cl}(\rho, r)) = f(\rho) \le \mathrm{sscl}(f(\rho), r).$$

Thus $f(\rho) = \mathrm{sscl}(f(\rho), r)$ and hence $f(\rho)$ is a fuzzy strongly r-semiclosed set of Y. Therefore f is fuzzy strongly r-semiclosed.

Theorem 4.9 *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a bijection and $r \in I_0$. Then f is a fuzzy strongly r-semiclosed map if and only if $f^{-1}(\mathrm{sscl}(\mu, r)) \le \mathrm{cl}(f^{-1}(\mu), r)$ for each fuzzy set μ of Y.*

Proof. Let f be a fuzzy strongly r-semiclosed map and μ a fuzzy set of Y. Then $f^{-1}(\mu)$ is a fuzzy set of X. Since f is onto,

$$\mathrm{sscl}(\mu, r) = \mathrm{sscl}(ff^{-1}(\mu), r) \le f(\mathrm{cl}(f^{-1}(\mu), r).$$

Since f is one-to-one, we have

$$\begin{aligned} f^{-1}(\mathrm{sscl}(\mu, r)) &\le f^{-1}f(\mathrm{cl}(f^{-1}(\mu), r)) \\ &= \mathrm{cl}(f^{-1}(\mu), r). \end{aligned}$$

Conversely, let ρ be a fuzzy r-closed set of X. Then $\mathrm{cl}(\rho, r) = \rho$. Since f is one-to-one,

$$f^{-1}(\mathrm{sscl}(f(\rho), r)) \le \mathrm{cl}(f^{-1}f(\rho), r) = \mathrm{cl}(\rho, r) = \rho.$$

Since f is onto, we have

$$f(\rho) \ge ff^{-1}(\mathrm{sscl}(f(\rho), r)) = \mathrm{sscl}(f(\rho), r) \ge f(\rho).$$

Thus $f(\rho) = \mathrm{sscl}(f(\rho), r)$ and hence $f(\rho)$ is a fuzzy strongly r-semiclosed set of Y. Therefore f is fuzzy strongly r-semiclosed.

Theorem 4.10 *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is fuzzy strongly r-semicontinuous (r-semiopen, r-semiclosed) if and only if $f : (X, \mathcal{T}_r) \to (Y, \mathcal{U}_r)$ is fuzzy strongly semicontinuous (semiopen, semiclosed).*

Proof. Straightforward.

Theorem 4.11 *Let $f : (X, T) \to (Y, U)$ be a map from a Chang's fuzzy topological space X to another Chang's fuzzy topological space Y and $r \in I_0$. Then f is fuzzy strongly semicontinuous (semiopen, semiclosed) if and only if $f : (X, T^r) \to (Y, U^r)$ is fuzzy strongly r-semicontinuous (r-semiopen, r-semiclosed).*

Proof. Straightforward.

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