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FUZZY ALMOST r-CONTINUOUS MAPS

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Abstract

In this paper, we introduce the notions of fuzzy r-regular open sets and fuzzy almost r-continuous maps, and investigate some of their basic properties.

Keywords: fuzzy r-regular open, fuzzy almost r-continuous

1. Introduction

As a generalization of sets, the concept of fuzzy sets was introduced by Zadeh. Chang [2] introduced fuzzy topological spaces and some other authors continued the investigation of such spaces. Some authors [4,5,7] introduced new definitions of fuzzy topology as a generalization of Chang's fuzzy topology. In this paper, we generalize the concepts of fuzzy regular open and fuzzy almost continuous of Azad [1]. We introduce the concepts of fuzzy r-regular open sets and fuzzy almost r-continuous maps, and then investigate some of their basic properties.

2. Preliminaries

In this paper, I will denote the unit interval $[0,1]$ of the real line and $I_0 = (0,1]$. A member μ of I^X is called a fuzzy set of X. For any $\mu \in I^X$, μ^c denotes the complement $1-\mu$. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on X with value 0 and 1, respectively. All other notations are standard notations of fuzzy set theory.

A *Chang's fuzzy topology* on X is a family T of fuzzy sets in X which satisfies the following three properties:

(1) $\tilde{0}, \tilde{1} \in T$.

(2) If $\mu_1, \mu_2 \in T$ then $\mu_1 \wedge \mu_2 \in T$.

(3) If $\mu_i \in T$ for each i, then $\bigvee \mu_i \in T$.

The pair (X, T) is called a *Chang's fuzzy topological space.*

A *fuzzy topology* on X is a map $\mathcal{T} : I^X \to I$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$,

(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$,

(3) $\mathcal{T}(\bigvee \mu_i) \geq \bigwedge \mathcal{T}(\mu_i)$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space.*

For each $\alpha \in (0,1]$, a *fuzzy point* x_α in X is a fuzzy set of X defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

In this case, x and α are called the *support* and the *value* of x_α, respectively. A fuzzy point x_α is said to *belong* to a fuzzy set μ of X, denoted by $x_\alpha \in \mu$, if $\alpha \leq \mu(x)$. A fuzzy point x_α in X is said to be *quasi-coincident* with μ, denoted by $x_\alpha q \mu$, if $\alpha + \mu(x) > 1$. A fuzzy set ρ of X is said to be *quasi-coincident* with a fuzzy set μ of X, denoted by $\rho q \mu$, if there is an $x \in X$ such that $\rho(x) + \mu(x) > 1$.

Definition 2.1 ([6]) Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is called

(1) a *fuzzy r-open set* of X if $\mathcal{T}(\mu) \geq r$,

(2) a *fuzzy r-closed set* of X if $\mathcal{T}(\mu^c) \geq r$.

Definition 2.2 ([3]) Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-closure* is defined by

$$\mathrm{cl}(\mu, r) = \bigwedge \{\rho \in I^X : \mu \leq \rho, \mathcal{T}(\rho^c) \geq r\}.$$

Definition 2.3 ([6]) Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-interior* is defined by

$$\operatorname{int}(\mu, r) = \bigvee\{\rho \in I^X : \mu \geq \rho, \mathcal{T}(\rho) \geq r\}.$$

Theorem 2.4 ([6]) *For a fuzzy set μ of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$, we have:*

(1) $\operatorname{int}(\mu, r)^c = \operatorname{cl}(\mu^c, r)$.

(2) $\operatorname{cl}(\mu, r)^c = \operatorname{int}(\mu^c, r)$.

Definition 2.5 ([6]) Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy r-semiopen* if there is a fuzzy r-open set ρ in X such that $\rho \leq \mu \leq \operatorname{cl}(\rho, r)$,

(2) *fuzzy r-semiclosed* if there is a fuzzy r-closed set ρ in X such that $\operatorname{int}(\rho, r) \leq \mu \leq \rho$.

Definition 2.6 ([6]) Let x_α be a fuzzy point of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then a fuzzy set μ of X is called

(1) a *fuzzy r-neighborhood* of x_α if there is a fuzzy r-open set ρ in X such that $x_\alpha \in \rho \leq \mu$,

(2) a *fuzzy r-quasi-neighborhood* of x_α if there is a fuzzy r-open set ρ in X such that $x_\alpha q \rho \leq \mu$.

Definition 2.7 ([6]) Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is called

(1) a *fuzzy r-continuous* map if $f^{-1}(\mu)$ is a fuzzy r-open set of X for each fuzzy r-open set μ of Y,

(2) a *fuzzy r-semicontinuous* map if $f^{-1}(\mu)$ is a fuzzy r-semiopen set of X for each fuzzy r-open set μ of Y,

(3) a *fuzzy r-irresolute* map if $f^{-1}(\mu)$ is a fuzzy r-semiopen set of X for each fuzzy r-semiopen set μ of Y.

3. Fuzzy r-regular open sets

We define the notions of fuzzy r-regular open sets and fuzzy r-regular closed sets, and investigate some of their properties.

Definition 3.1 Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy r-regular open* if $\operatorname{int}(\operatorname{cl}(\mu, r), r) = \mu$,

(2) *fuzzy r-regular closed* if $\operatorname{cl}(\operatorname{int}(\mu, r), r) = \mu$.

Theorem 3.2 *Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is fuzzy r-regular open if and only if μ^c is fuzzy r-regular closed.*

Proof. It follows from Theorem 2.4.

Remark 3.3 Clearly, every fuzzy r-regular open (r-regular closed) set is fuzzy r-open (r-closed). That the converse need not be true is shown by the following example. The example also shows that the union (intersection) of any two fuzzy r-regular open (r-regular closed) sets need not be fuzzy r-regular open (r-regular closed).

Example 3.4 Let $X = I$ and μ_1, μ_2 and μ_3 be fuzzy sets of X defined by

$$\mu_1(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1; \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{4}, \\ -4x + 2 & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1; \end{cases}$$

and

$$\mu_3(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{4}, \\ \frac{1}{3}(4x - 1) & \text{if } \frac{1}{4} \leq x \leq 1. \end{cases}$$

Define $\mathcal{T} : I^X \to I$ by

$$\mathcal{T}(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \mu_2, \mu_1 \vee \mu_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $\mathcal{T}$ is a fuzzy topology on X.

(1) Clearly, $\mu_1 \vee \mu_2$ is fuzzy $\frac{1}{2}$-open. Since $\operatorname{int}(\operatorname{cl}(\mu_1 \vee \mu_2, \frac{1}{2}), \frac{1}{2}) = \tilde{1} \neq \mu_1 \vee \mu_2$, $\mu_1 \vee \mu_2$ is not a fuzzy $\frac{1}{2}$-regular open set.

(2) Since $\operatorname{int}(\operatorname{cl}(\mu_1, \frac{1}{2}), \frac{1}{2}) = \operatorname{int}(\mu_2^c, \frac{1}{2}) = \mu_1$ and $\operatorname{int}(\operatorname{cl}(\mu_2, \frac{1}{2}), \frac{1}{2}) = \operatorname{int}(\mu_1^c, \frac{1}{2}) = \mu_2$, μ_1 and μ_2 are fuzzy $\frac{1}{2}$-regular open sets. But $\mu_1 \vee \mu_2$ is not a fuzzy $\frac{1}{2}$-regular open set.

(3) In view of Theorem 3.2, μ_1^c and μ_2^c are fuzzy $\frac{1}{2}$-regular closed sets but $\mu_1^c \wedge \mu_2^c = (\mu_1 \vee \mu_2)^c$ is not a fuzzy $\frac{1}{2}$-regular closed set.

Theorem 3.5 (1) *The intersection of two fuzzy r-regular open sets is fuzzy r-regular open.*

(2) *The union of two fuzzy r-regular closed sets is fuzzy r-regular closed.*

Proof. (1) Let μ and ρ be any two fuzzy r-regular open sets of a fuzzy topological space X.

Then μ and ρ are fuzzy r-open sets and hence $\mathcal{T}(\mu \wedge \rho) \geq \mathcal{T}(\mu) \wedge \mathcal{T}(\rho) \geq r$. Thus $\mu \wedge \rho$ is a fuzzy r-open set. Since $\mu \wedge \rho \leq \mathrm{cl}(\mu \wedge \rho, r)$,

$$\mathrm{int}(\mathrm{cl}(\mu \wedge \rho, r), r) \geq \mathrm{int}(\mu \wedge \rho, r) = \mu \wedge \rho.$$

Now, $\mu \wedge \rho \leq \mu$ and $\mu \wedge \rho \leq \rho$ implies

$$\mathrm{int}(\mathrm{cl}(\mu \wedge \rho, r), r) \leq \mathrm{int}(\mathrm{cl}(\mu, r), r) = \mu$$

and

$$\mathrm{int}(\mathrm{cl}(\mu \wedge \rho, r), r) \leq \mathrm{int}(\mathrm{cl}(\rho, r), r) = \rho.$$

Hence $\mathrm{int}(\mathrm{cl}(\mu \wedge \rho, r), r) \leq \mu \wedge \rho$. Therefore $\mu \wedge \rho$ is fuzzy r-regular open.

(2) It follows from (1) using Theorem 3.2.

Theorem 3.6 (1) *The fuzzy r-closure of a fuzzy r-open set is fuzzy r-regular closed.*

(2) *The fuzzy r-interior of a fuzzy r-closed set is fuzzy r-regular open.*

Proof. (1) Let μ be a fuzzy r-open set of a fuzzy topological space X. Then clearly $\mathrm{int}(\mathrm{cl}(\mu, r), r) \leq \mathrm{cl}(\mu, r)$ implies that

$$\mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mu, r), r), r) \leq \mathrm{cl}(\mathrm{cl}(\mu, r), r) = \mathrm{cl}(\mu, r).$$

Since μ is fuzzy r-open, $\mu = \mathrm{int}(\mu, r)$. Also since $\mu \leq \mathrm{cl}(\mu, r)$, $\mu = \mathrm{int}(\mu, r) \leq \mathrm{int}(\mathrm{cl}(\mu, r), r)$. Thus $\mathrm{cl}(\mu, r) \leq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mu, r), r), r)$. Hence $\mathrm{cl}(\mu, r)$ is a fuzzy r-regular closed set.

(2) Similar to (1).

Theorem 3.7 *Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is fuzzy r-regular open (r-regular closed) in $(X, \mathcal{T})$ if and only if μ is fuzzy regular open (regular closed) in $(X, \mathcal{T}_r)$.*

Proof. Straightforward.

Theorem 3.8 *Let μ be a fuzzy set of a Chang's fuzzy topological space (X, T) and $r \in I_0$. Then μ is a fuzzy regular open (regular closed) in (X, T) if and only if μ is fuzzy r-regular open (r-regular closed) in (X, T^r).*

Proof. Straightforward.

4. Fuzzy almost r-continuous maps

We are going to introduce the notions of fuzzy almost r-continuous maps and investigate some of their properties. Also, we observe the relations between fuzzy almost r-continuous maps, fuzzy r-continuous maps and fuzzy r-semicontinuous maps.

Definition 4.1 Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is called

(1) a *fuzzy almost r-continuous* map if $f^{-1}(\mu)$ is a fuzzy r-open set of X for each fuzzy r-regular open set μ of Y, or equivalently, $f^{-1}(\mu)$ is a fuzzy r-closed set of X for each fuzzy r-regular closed set μ of Y,

(2) a *fuzzy almost r-open* map if $f(\rho)$ is a fuzzy r-open set of Y for each fuzzy r-regular open set ρ of X,

(3) a *fuzzy almost r-closed* map if $f(\rho)$ is a fuzzy r-closed set of Y for each fuzzy r-regular closed set ρ of X.

Theorem 4.2 *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *f is a fuzzy almost r-continuous map.*

(2) $f^{-1}(\mu) \leq \mathrm{int}(f^{-1}(\mathrm{int}(\mathrm{cl}(\mu, r), r)), r)$ *for each fuzzy r-open set μ of Y.*

(3) $\mathrm{cl}(f^{-1}(\mathrm{cl}(\mathrm{int}(\mu, r), r)), r) \leq f^{-1}(\mu)$ *for each fuzzy r-closed set μ of Y.*

Proof. (1) $\Rightarrow$ (2) Let f be fuzzy almost r-continuous and μ any fuzzy r-open set of Y. Then

$$\mu = \mathrm{int}(\mu, r) \leq \mathrm{int}(\mathrm{cl}(\mu, r), r).$$

By Theorem 3.6(2), $\mathrm{int}(\mathrm{cl}(\mu, r), r)$ is a fuzzy r-regular open set of Y. Since f is fuzzy almost r-continuous, $f^{-1}(\mathrm{int}(\mathrm{cl}(\mu, r), r))$ is a fuzzy r-open set of X. Hence

$$\begin{aligned} f^{-1}(\mu) &\leq f^{-1}(\mathrm{int}(\mathrm{cl}(\mu, r), r)) \\ &= \mathrm{int}(f^{-1}(\mathrm{int}(\mathrm{cl}(\mu, r), r)), r). \end{aligned}$$

(2) $\Rightarrow$ (3) Let μ be a fuzzy r-closed set of Y. Then μ^c is a fuzzy r-open set of Y. By (2),

$$f^{-1}(\mu^c) \leq \mathrm{int}(f^{-1}(\mathrm{int}(\mathrm{cl}(\mu^c, r), r)), r).$$

Hence

$$\begin{aligned} f^{-1}(\mu) &= f^{-1}(\mu^c)^c \\ &\geq \mathrm{int}(f^{-1}(\mathrm{int}(\mathrm{cl}(\mu^c, r), r)), r)^c \\ &= \mathrm{cl}(f^{-1}(\mathrm{cl}(\mathrm{int}(\mu, r), r)), r). \end{aligned}$$

(3) ⇒ (1) Let μ be a fuzzy r-regular closed set of Y. Then μ is a fuzzy r-closed set of Y and hence

$$\begin{aligned} f^{-1}(\mu) &\geq \mathrm{cl}(f^{-1}(\mathrm{cl}(\mathrm{int}(\mu, r), r)), r) \\ &= \mathrm{cl}(f^{-1}(\mu), r). \end{aligned}$$

Thus $f^{-1}(\mu) = \mathrm{cl}(f^{-1}(\mu), r)$ and hence $f^{-1}(\mu)$ is a fuzzy r-closed set of X. Therefore f is a fuzzy almost r-continuous map.

Theorem 4.3 *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then f is fuzzy almost r-open if and only if $f(\mathrm{int}(\rho, r)) \leq \mathrm{int}(f(\rho), r)$ for each fuzzy r-semiclosed set ρ of X.*

Proof. Let f be fuzzy almost r-open and ρ a fuzzy r-semiclosed set of X. Then $\mathrm{int}(\rho, r) \leq \mathrm{int}(\mathrm{cl}(\rho, r), r) \leq \rho$. Note that $\mathrm{cl}(\rho, r)$ is a fuzzy r-closed set of X. By Theorem 3.6(2), $\mathrm{int}(\mathrm{cl}(\rho, r), r)$ is a fuzzy r-regular open set of X. Since f is fuzzy almost r-open, $f(\mathrm{int}(\mathrm{cl}(\rho, r), r))$ is a fuzzy r-open set of X. Thus we have

$$\begin{aligned} f(\mathrm{int}(\rho, r)) &\leq f(\mathrm{int}(\mathrm{cl}(\rho, r), r)) \\ &= \mathrm{int}(f(\mathrm{int}(\mathrm{cl}(\rho, r), r)), r) \\ &\leq \mathrm{int}(f(\rho), r). \end{aligned}$$

Conversely, let ρ be a fuzzy r-regular open set of X. Then ρ is fuzzy r-open and hence $\mathrm{int}(\rho, r) = \rho$. Since $\mathrm{int}(\mathrm{cl}(\rho, r), r) = \rho$, ρ is fuzzy r-semiclosed. So

$$f(\rho) = f(\mathrm{int}(\rho, r)) \leq \mathrm{int}(f(\rho), r) \leq f(\rho).$$

Thus $f(\rho) = \mathrm{int}(f(\rho), r)$ and hence $f(\rho)$ is a fuzzy r-open set of Y.

The global property of fuzzy almost r-continuity can be rephrased to the local property in terms of neighborhood and quasi-neighborhood as following two theorems.

Theorem 4.4 *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then f is fuzzy almost r-continuous if and only if for every fuzzy point x_α in X and every fuzzy r-neighborhood μ of $f(x_\alpha)$, there is a fuzzy r-neighborhood ρ of x_α such that $x_\alpha \in \rho$ and $f(\rho) \leq \mathrm{int}(\mathrm{cl}(\mu, r), r)$.*

Proof. Let x_α be a fuzzy point in X and μ a fuzzy r-neighborhood of $f(x_\alpha)$. Then there is a fuzzy r-open set λ of Y such that $f(x_\alpha) \in \lambda \leq \mu$. So $x_\alpha \in f^{-1}(\lambda) \leq f^{-1}(\mu)$. Since f is fuzzy almost r-continuous,

$$\begin{aligned} f^{-1}(\lambda) &\leq \mathrm{int}(f^{-1}(\mathrm{int}(\mathrm{cl}(\lambda, r), r)), r) \\ &\leq \mathrm{int}(f^{-1}(\mathrm{int}(\mathrm{cl}(\mu, r), r)), r). \end{aligned}$$

Put $\rho = f^{-1}(\mathrm{int}(\mathrm{cl}(\mu, r), r))$. Then $x_\alpha \in f^{-1}(\lambda) \leq \mathrm{int}(\rho, r) \leq \rho$. By Theorem 3.6(2), $\mathrm{int}(\mathrm{cl}(\mu, r), r)$ is fuzzy r-regular open. Since f is fuzzy almost r-continuous, $\rho = f^{-1}(\mathrm{int}(\mathrm{cl}(\mu, r), r))$ is fuzzy r-open. Thus ρ is a fuzzy r-neighborhood of x_α and

$$f(\rho) = ff^{-1}(\mathrm{int}(\mathrm{cl}(\mu, r), r)) \leq \mathrm{int}(\mathrm{cl}(\mu, r), r).$$

Conversely, let μ be a fuzzy r-regular open set of Y and $x_\alpha \in f^{-1}(\mu)$. Then μ is fuzzy r-open and hence μ is a fuzzy r-neighborhood of $f(x_\alpha)$. By hypothesis, there is a fuzzy r-neighborhood ρ_{x_α} of x_α such that $x_\alpha \in \rho_{x_\alpha}$ and $f(\rho_{x_\alpha}) \leq \mathrm{int}(\mathrm{cl}(\mu, r), r) = \mu$. Since ρ_{x_α} is a fuzzy r-neighborhood of x_α, there is a fuzzy r-open set λ_{x_α} of X such that

$$x_\alpha \in \lambda_{x_\alpha} \leq \rho_{x_\alpha} \leq f^{-1}f(\rho_{x_\alpha}) \leq f^{-1}(\mu).$$

So we have

$$\begin{aligned} f^{-1}(\mu) &= \bigvee\{x_\alpha : x_\alpha \in f^{-1}(\mu)\} \\ &\leq \bigvee\{\lambda_{x_\alpha} : x_\alpha \in f^{-1}(\mu)\} \\ &\leq f^{-1}(\mu). \end{aligned}$$

Thus $f^{-1}(\mu) = \bigvee\{\lambda_{x_\alpha} : x_\alpha \in f^{-1}(\mu)\}$ is fuzzy r-open in X and hence f is almost r-continuous.

Theorem 4.5 *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map and $r \in I_0$. Then f is a fuzzy almost r-continuous map if and only if for every fuzzy point x_α in X and every fuzzy r-quasi-neighborhood μ of $f(x_\alpha)$, there is a fuzzy r-quasi-neighborhood ρ of x_α such that $x_\alpha q \rho$ and $f(\rho) \leq \mathrm{int}(\mathrm{cl}(\mu, r), r)$.*

Proof. Let x_α be a fuzzy point in X and μ a fuzzy r-quasi-neighborhood of $f(x_\alpha)$. Then there is a fuzzy r-open set λ in Y such that $f(x_\alpha) q \lambda \leq \mu$. So $x_\alpha q f^{-1}(\lambda)$. Since f is fuzzy almost r-continuous,

$$\begin{aligned} f^{-1}(\lambda) &\leq \mathrm{int}(f^{-1}(\mathrm{int}(\mathrm{cl}(\lambda, r), r)), r) \\ &\leq \mathrm{int}(f^{-1}(\mathrm{int}(\mathrm{cl}(\mu, r), r)), r). \end{aligned}$$

Put $\rho = f^{-1}(\mathrm{int}(\mathrm{cl}(\mu, r), r))$. Then $x_\alpha q f^{-1}(\lambda) \leq \mathrm{int}(\rho, r) \leq \rho$. So $x_\alpha q \rho$. Since $\mathrm{int}(\mathrm{cl}(\mu, r), r)$ is fuzzy r-regular open and f is fuzzy almost r-continuous, $\rho = f^{-1}(\mathrm{int}(\mathrm{cl}(\mu, r), r))$ is fuzzy r-open. Thus ρ is a fuzzy r-quasi-neighborhood of x_α and

$$f(\rho) = ff^{-1}(\mathrm{int}(\mathrm{cl}(\mu, r), r)) \leq \mathrm{int}(\mathrm{cl}(\mu, r), r).$$

Conversely, let μ be a fuzzy r-regular open set of Y. If $f^{-1}(\mu) = \tilde{0}$, then it is obvious. Suppose x_α is a fuzzy point in $f^{-1}(\mu)$ such that $\alpha < f^{-1}(\mu)(x)$. Then $\alpha < \mu(f(x))$ and hence $f(x)_{1-\alpha} q \mu$. So μ is a fuzzy r-quasi-neighborhood of $f(x)_{1-\alpha} =$

$f(x_{1-\alpha})$. By hypothesis, there is a fuzzy r-quasi-neighborhood ρ_{x_α} of $x_{1-\alpha}$ such that $x_{1-\alpha}q\rho_{x_\alpha}$ and $f(\rho_{x_\alpha}) \leq \mathrm{int}(\mathrm{cl}(\mu, r), r) = \mu$. Since ρ_{x_α} is a fuzzy r-quasi-neighborhood of $x_{1-\alpha}$, there is a fuzzy r-open set λ_{x_α} in X such that

$$x_{1-\alpha}q\lambda_{x_\alpha} \leq \rho_{x_\alpha} \leq f^{-1}f(\rho_{x_\alpha}) \leq f^{-1}(\mu).$$

Then $\alpha < \lambda_{x_\alpha}(x)$ and hence $x_\alpha \in \lambda_{x_\alpha}$. So

$$\begin{aligned} f^{-1}(\mu) &= \bigvee\{x_\alpha : x_\alpha \text{ is a fuzzy point in } f^{-1}(\mu) \\ &\qquad \text{such that } \alpha < f^{-1}(\mu)(x)\} \\ &\leq \bigvee\{\lambda_{x_\alpha} : x_\alpha \text{ is a fuzzy point in } f^{-1}(\mu) \\ &\qquad \text{such that } \alpha < f^{-1}(\mu)(x)\} \\ &\leq f^{-1}(\mu) \end{aligned}$$

and hence

$$\begin{aligned} f^{-1}(\mu) &= \bigvee\{\lambda_{x_\alpha} : x_\alpha \text{ is a fuzzy point in } f^{-1}(\mu) \\ &\qquad \text{such that } \alpha < f^{-1}(\mu)(x)\}. \end{aligned}$$

Thus $f^{-1}(\mu)$ is fuzzy r-open in X. Therefore f is fuzzy almost r-continuous.

Theorem 4.6 *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be fuzzy r-semicontinuous and fuzzy almost r-open. Then f is fuzzy r-irresolute.*

Proof. Let μ be fuzzy r-semiclosed in Y. Then $\mathrm{int}(\mathrm{cl}(\mu, r), r) \leq \mu$. Since f is fuzzy r-semicontinuous,

$$\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r) \leq f^{-1}(\mathrm{cl}(\mu, r)).$$

Thus we have

$$\begin{aligned} \mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r) &= \mathrm{int}(\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r), r) \\ &\leq \mathrm{int}(f^{-1}(\mathrm{cl}(\mu, r)), r). \end{aligned}$$

Since f is fuzzy r-semicontinuous and $\mathrm{cl}(\mu, r)$ is fuzzy r-closed, $f^{-1}(\mathrm{cl}(\mu, r))$ is a fuzzy r-semiclosed set of X. Since f is fuzzy almost r-open,

$$\begin{aligned} f(\mathrm{int}(f^{-1}(\mathrm{cl}(\mu, r)), r) &\leq \mathrm{int}(ff^{-1}(\mathrm{cl}(\mu, r)), r) \\ &\leq \mathrm{int}(\mathrm{cl}(\mu, r), r) \\ &\leq \mu. \end{aligned}$$

Hence we have

$$\begin{aligned} \mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r) &\leq f^{-1}f(\mathrm{int}(\mathrm{cl}(f^{-1}(\mu), r), r) \\ &\leq f^{-1}f(\mathrm{int}(f^{-1}(\mathrm{cl}(\mu, r)), r)) \\ &\leq f^{-1}(\mu). \end{aligned}$$

Thus $f^{-1}(\mu)$ is fuzzy r-semiclosed in X and hence f is fuzzy r-irresolute.

Remark 4.7 Clearly a fuzzy r-continuous map is a fuzzy almost r-continuous map. That the converse need not be true is shown by the following example. Also, the example shows that a fuzzy almost r-continuous map need not be a fuzzy r-semicontinuous map.

Example 4.8 Let $X = I$ and μ_1, μ_2 and μ_3 be fuzzy sets of X defined by

$$\mu_1(x) = x;$$

$$\mu_2(x) = 1 - x;$$

and

$$\mu_3(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Define $\mathcal{T}_1 : I^X \to I$ and $\mathcal{T}_2 : I^X \to I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1} \\ \frac{1}{2} & \text{if } \mu = \mu_1, \mu_2, \mu_1 \vee \mu_2, \mu_1 \wedge \mu_2 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathcal{T}_2(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1} \\ \frac{1}{2} & \text{if } \mu = \mu_1, \mu_2, \mu_3, \mu_1 \vee \mu_2, \mu_1 \wedge \mu_2 \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $\mathcal{T}_1, \mathcal{T}_2$ are fuzzy topologies on X. Consider the identity map $1_X : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$. It is clear that $\mu_1, \mu_2, \mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ are fuzzy $\frac{1}{2}$-regular open of $(X, \mathcal{T}_2)$ while μ_3 is not. Noting that $\mathcal{T}_1(\mu_3) = 0$, it is obvious that 1_X is a fuzzy $\frac{1}{2}$-almost continuous map which is not a fuzzy $\frac{1}{2}$-continuous map. Also, because $\tilde{0}$ is the only fuzzy $\frac{1}{2}$-open set contained in μ_3, $\mu_3 = 1_X^{-1}(\mu_3)$ is not a fuzzy $\frac{1}{2}$-semiopen set of $(X, \mathcal{T}_1)$ and hence 1_X is not a fuzzy $\frac{1}{2}$-semicontinuous map.

Example 4.9 *A fuzzy r-semicontinuous map need not be a fuzzy almost r-continuous map.*

Let $(X, \mathcal{T})$ be a fuzzy topological space as described in Example 3.4 and let $f : (X, \mathcal{T}) \to (X, \mathcal{T})$ be defined by $f(x) = \frac{x}{2}$. Simple computations give $f^{-1}(\tilde{0}) = \tilde{0}, f^{-1}(\tilde{1}) = \tilde{1}, f^{-1}(\mu_1) = \tilde{0}$ and $f^{-1}(\mu_2) = \mu_1^c = f^{-1}(\mu_1 \vee \mu_2)$. Since $\mathrm{cl}(\mu_2, \frac{1}{2}) = \mu_1^c$, μ_1^c is a fuzzy $\frac{1}{2}$-semiopen set and hence f is a fuzzy $\frac{1}{2}$-semicontinuous map. But $f^{-1}(\mu_2) = \mu_1^c$ and

$$\begin{aligned} &\mathrm{int}(f^{-1}(\mathrm{int}(\mathrm{cl}(\mu_2, \tfrac{1}{2}), \tfrac{1}{2})), \tfrac{1}{2}) \\ =\ &\mathrm{int}(f^{-1}(\mathrm{int}(\mu_1^c, \tfrac{1}{2})), \tfrac{1}{2}) \\ =\ &\mathrm{int}(f^{-1}(\mu_2), \tfrac{1}{2}) = \mathrm{int}(\mu_1^c, \tfrac{1}{2}) = \mu_2. \end{aligned}$$

Thus $f^{-1}(\mu_2) \not\leq \mathrm{int}(f^{-1}(\mathrm{int}(\mathrm{cl}(\mu_2, \frac{1}{2}), \frac{1}{2})), \frac{1}{2})$ and hence f is not a fuzzy almost $\frac{1}{2}$-continuous map.

From Example 4.8 and 4.9 we have the following result.

Theorem 4.10 *Fuzzy r-semicontinuity and fuzzy almost r-continuity are independent notions.*

Definition 4.11 Let $(X, \mathcal{T})$ be a fuzzy topological space and $r \in I_0$. Then $(X, \mathcal{T})$ is called a *fuzzy r-semiregular space* if each fuzzy r-open set of X is a union of fuzzy r-regular open sets.

Theorem 4.12 *Let $r \in I_0$ and $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to a fuzzy r-semiregular space Y. Then f is fuzzy almost r-continuous if and only if f is fuzzy r-continuous.*

Proof. Due to Remark 4.7, it suffices to show that if f is fuzzy almost r-continuous then it is fuzzy r-continuous. Let μ be a fuzzy r-open set of Y. Since $(Y, \mathcal{U})$ is a r-semiregular space, $\mu = \bigvee \mu_i$, where μ_i's are fuzzy r-regular open sets of Y. Then since f is a fuzzy almost r-continuous map, $f^{-1}(\mu_i)$ is a fuzzy r-open set for each i. So

$$\begin{aligned} \mathcal{T}(f^{-1}(\mu)) &= \mathcal{T}(f^{-1}(\bigvee \mu_i)) = \mathcal{T}(\bigvee f^{-1}(\mu_i)) \\ &\geq \bigwedge \mathcal{T}(f^{-1}(\mu_i)) \geq r. \end{aligned}$$

Thus $f^{-1}(\mu)$ is fuzzy r-open of X and hence f is a fuzzy r-continuous map.

Theorem 4.13 *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is fuzzy almost r-continuous (r-open, r-closed) if and only if $f : (X, \mathcal{T}_r) \to (Y, \mathcal{U}_r)$ is fuzzy almost continuous (open, closed).*

Proof. Straightforward.

Theorem 4.14 *Let $f : (X, T) \to (Y, U)$ be a map from a Chang's fuzzy topological space X to another Chang's fuzzy topological space Y and $r \in I_0$. Then f is fuzzy almost continuous (open, closed) if and only if $f : (X, T^r) \to (Y, U^r)$ is fuzzy almost r-continuous (r-open, r-closed).*

Proof. Straightforward.

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