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COMBINING EQUAL-LIFE MULTILEVEL INVESTMENTS USING FUZZY DYNAMIC PROGRAMMING

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Abstract: Dynamic programming is applicable to any situation where items from several groups must be combined to form an entity, such as a composite investment or a transportation route connecting several districts. The most desirable entity is constructed in stages by forming sub-entities that are candidates for inclusion in the most desirable entity are retained, and all other sub-entities are discarded. In the paper, the fuzzy dynamic programming is applied to the situation where each investment in the set has the following characteristics: the amount to be invested has several possible values, and the rate of return varies with the amount invested. Each sum that may be invested represents a distinct level of investment, and the investment therefore has multiple levels. A numeric example constructing a combination of multilevel investments is given in the paper.
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1. Introduction

Traditional dynamic programming is a technique well known in operations research and used to solve optimization problems that can be composed into subproblems of one decision-variable each. The idea underlying dynamic programming is to view the problem as a multistage decision process, the optimal policy to which can be determined recursively. The problem is solved by solving recursively the following [1]:

$$\max_{d_i} R_i(x_i,d_i) = \max_{d_i} r_i(x_i,d_i) \circ R_{i+1}(x_{i+1}) \qquad (1)$$

such that

$$x_{i+1} = t_i(x_i,d_i)$$
$$i = 1,2,\ldots,N-1$$

where

x_i: state variables

d_i: decision variables

$r_i(x_i,d_i)$: stage rewards

$R_i(d_N,\ldots,d_{N-i},x_N)$: a reward function

$t_i(d_i,x_i)$: a transformation function

or

$$\max_{d_i} R_i(x_i,d_i) = \max_{d_i}\{r_i(x_i,d_i) \circ R_{i+1}(t_i(x_i,d_i))\} \qquad (2)$$

All variables, rewards, and transformations are supposed to be crisp.

Many capital budgeting problems allow of a *dynamic* formulation. There may actually be several decision points, but even if this is not so if the decision problem can be divided up into *stages* than a discrete dynamic expression is possible. Many problems allow of either static or dynamic expression. The choice of form would be up to the problem solver. Characteristically, a dynamic economizing model allocates scarce resources between alternative uses between initial and terminal times.

In the case of equal-life multilevel investments, each investment in the set has the following characteristic: the amount to be invested has several *possible* values, and the rate of return varies with the amount invested. Each sum that may be invested represents a distinct *level* of investment, and the investment therefore has multiple levels. Examples of multilevel investments may be the purchase of labor-saving equipment where several types of equipment are available and each type has a unique cost. The level of investment in labor-saving equipment depends on the type of equipment selected. Another example is the construction and rental of an office building, where the owner-builder has a choice concerning the number of stories the building is to contain [2].

2. Fuzzy Dynamic Programming

Bellman and Zadeh [3] suggested for the first time a fuzzy approach to this type of problem. They use the following terms to define the fuzzy dynamic programming: $\widetilde{X}_i \in \widetilde{X}, i = 1,2,\ldots,N$: crisp state variable where $\widetilde{X} = \{\tau_1,\ldots,\tau_N\}$ is the set of values permitted for the state variables; $d_i \in \widetilde{D}, i = 1,2,\ldots,N$: crisp decision variable where $\widetilde{D} = \{\alpha_1,\ldots,\alpha_m\}$ is the set of possible decisions.

$$x_{i+1} = t(x_i, d_i)\text{: crisp transformation function}$$

For each stage t, $t=0,1,\ldots,N-1$, we define:

1. A fuzzy constraint $\widetilde{C}_t$ limiting the decision space and characterized by its membership function

$$\mu_{\widetilde{C}}(d_t)$$

2. A fuzzy goal $\widetilde{G}_N$ characterized by the membership function

$$\mu_{\widetilde{G}_N}(x_N)$$

The problem is to determine the maximizing decision

$$\widetilde{D}^0 = \{d_i^0\}, i = 0,1,2,\ldots,N, \text{ for a given } x_0 \quad (3)$$

The fuzzy set decision is the *confluence* of the constraints and the goal(s), that is,

$$\widetilde{D} = \bigcap_{t=0}^{N-1} \widetilde{C}_t \cap \widetilde{G}_N \quad (4)$$

Using the min-operator for the aggregation of the fuzzy constraints and the goal, the membership function of the fuzzy set decision is

$$\mu_{\widetilde{D}}(d_0,\ldots,d_{N-1}) = \min\{\mu_{\widetilde{C}_0}(d_0),\ldots,\mu_{\widetilde{C}_{N-1}}(d_{N-1}),\mu_{\widetilde{G}_N}(x_N)\} \quad (5)$$

The membership function of the maximizing decision is then

$$\mu_{\widetilde{D}^0}(d_0^0,\ldots,d_{N-1}^0) = \max_{d_{01}\ldots d_{N-2}} \max_{d_{N-1}} [\min\{\mu_{\widetilde{C}_0}(d_0),\ldots,\mu_{\widetilde{G}_N}(t_N(x_{N-1},d_{N-1}))\}] \quad (6)$$

where d_i^0 denotes the optimal decision on stage i. If K is a constant and g is any function of d_{N-1}, we can write

$$\max_{d_{N-1}} \min\{g(d_{N-1}),K\} = \min\{K, \max_{d_{N-1}} g(d_{N-1})\} \quad (7)$$

and Eq. (6) can be expressed as

$$\mu_{\widetilde{D}^0}(d_0^0,\ldots,d_{N-1}^0) = \max_{d_0,\ldots,d_{N-1}} \min\{\mu_{\widetilde{C}_0}(d_0),\ldots,\mu_{\widetilde{G}_{N-1}}(x_{N-1})\} \quad (8)$$

with

$$\mu_{\widetilde{G}_{N-1}}(x_{N-1}) = \max_{d_{N-1}} \min\{\mu_{\widetilde{C}_{N-1}}(d_{N-1}),\mu_{\widetilde{G}_N}(t_N(x_{N-1},d_{N-1}))\} \quad (9)$$

$\widetilde{D}$ is thus determined recursively.

3. A Numeric Example

A firm has $ 600,000 available for investment, and three investment plans A, B, and C are under consideration. Each plan has these features: the amount that can be invested is a multiple of $ 100,000; the investors receive annual dividends; capital is recovered when the venture terminates at the end of 5 years. Table 1 lists the annual dividends corresponding to the various levels of investment. Devise the most lucrative composite investment [2].

Table 1. The Annual Dividends of *A, B, C* (x $ 1,000)

	Dividends		
Investment	Plan *A*	Plan *B*	Plan *C*
100	25	10	15
200	44	32	31
300	63	60	48
400	80	91	56
500	89	93	79
600	95	94	102

First, the problem will be solved by crisp dynamic programming. The solution consists of the following steps:

1. Devise all possible investments that encompass plans A and B alone, applying an upper limit of $600,000 to the amount invested. Compound the corresponding annual dividends. Let Q denote the amount of capital to be allocated to the combination of plans A and B, where Q can range from $ 100,000 to $ 600,000. Although both plans A and B fall within our purview in this step, it is understood that Q can be allocated to A alone or to B alone. Table 2. displays all possible combinations corresponding to every possible value of Q, together with their respective dividends.

Table 2. Combinations of Plans A and B (x $ 1,000)

Total Investment, Q	A	B	Annual Dividend
600	600	0	95 + 0 = 95
	500	100	86 + 10 = 99
	400	200	80 + 32 = 112
	300	300	63 + 60 = 123
	200	400	44 + 91 = 135*
	100	500	25 + 93 = 118
	0	600	0 + 94 = 94
500	500	0	89 + 0 = 89
	400	100	80 + 10 = 90
	300	200	63 + 32 = 95
	200	300	44 + 60 = 104
	100	400	25 + 91 = 116*
	0	500	0 + 93 = 93
400	400	0	80 + 0 = 80
	300	100	63 + 10 = 73
	200	200	44 + 32 = 76
	100	300	25 + 60 = 85
	0	400	0 + 91 = 91*
300	300	0	63 + 0 = 63*
	200	100	44 + 10 = 54
	100	200	25 + 32 = 57
	0	300	0 + 60 = 60
200	200	0	44 + 0 = 44*
	100	100	25 + 10 = 35
	0	200	0 + 32 = 32
100	100	0	25 + 0 = 25*
	0	100	0 + 10 = 10

2. Identify the most lucrative combination of Plans A and B corresponding to every possible value of Q. In Table 2, the most lucrative combinations are identified by asterisks.

3. Devise all possible investments that encompass plans A, B, and C, and identify the most lucrative one. Table 3 gives the possible investments and their respective dividends. The most lucrative composite investment that encompasses all three plans is the one in which $ 200,000 is placed in Plan A, $ 400,000 in Plan B, and nothing in Plan C.

Table 3.Combination of Plans A, B, and C (x $ 1,000)

Combinations of A and B	C	Annual Dividend
600	0	135 + 0 = 135
500	100	116 + 15 = 131
400	200	91 + 31 = 122
300	300	63 + 48 = 111
200	400	44 + 56 = 100
100	500	25 + 79 = 104
0	600	0 + 102 = 102

In the case of fuzziness, dividends are assumed to be given together with their possibility values. Table 4 shows the fuzzy dividend of each plan:

Table 4. Fuzzy Annual Dividends of A, B, and C (x $ 1,000)

Investment	A
100	{(30, 0.6), (25, 1.0)}
200	{(47, 0.8), (44, 1.0)}
300	{(76, 0.6), (63, 1.0)}
400	{(99, 0.7), (80, 1.0)}
500	{(95, 0.8), (89, 1.0)}
600	{(96, 0.7), (95, 1.0)}
Investment	**B**
100	{(14, 0.6), (10, 1.0)}
200	{(44, 0.8), (32, 1.0)}
300	{(73, 0.7), (60, 1.0)}
400	{(94, 0.9), (91, 1.0)}
500	{(96, 0.7), (93, 1.0)}
600	{(98, 0.8), (94, 1.0)}
Investment	**C**
100	{(17, 0.8), (15, 1.0)}
200	{(37, 0.7), (31, 1.0)}
300	{(58, 0.6), (48, 1.0)}
400	{(61, 0.8), (56, 1.0)}
500	{(81, 0.9), (79, 1.0)}
600	{(104, 0.8), (102, 1.0)}

<u>For the total investment of $ 600,000 in A and B:</u>

Investment in A: $ 600,000
B: $ 0 : $\tilde{C}_0^A \cup \tilde{C}_0^B = \{(96,0.7),(95,1.0)\}$

Investment in *A*: $ 500,000
B: $ 100,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(109,0.8),(99,1.0)}

Investment in A: $ 400,000
B: $ 200,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(143,0.8),(112,1.0)}

Investment in A: $ 300,000
B: $ 300,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(149,0.7),(123,1.0)}

Investment in A: $ 200,000
B: $ 400,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(141,0.9),(135,1.0)}*

Investment in A: $ 100,000
B: $ 500,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(126,0.7),(118,1.0)}

Investment in A: $ 0
B: $ 600,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(98,0.8),(94,1.0)}

<u>For the total investment of $ 500,000 in A and B:</u>

Investment in A: $ 500,000
B: $ 0: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(95,0.8),(89,1.0)}

Investment in A: $ 400,000
B: $ 100,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(113,0.7),(90,1.0)}

Investment in A: $ 300,000
B : $ 200,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(120,0.8),(95,1.0)}

Investment in A: $ 200,000
B: $ 300,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(120,0.8),(104,1.0)}

Investment in A: $ 100,000
B: $ 400,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(124,0.9),(116,1.0)}*

Investment in A: $ 0
B: $ 500,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(96,0.7),(93,1.0)}

<u>For the total investment of $ 400,000 in A and B:</u>

Investment in A: $ 400,000
B: $ 0: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(99,0.7),(80,1.0)}

Investment in A: $ 300,000
B: $ 100,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(90,0.6),(73,1.0)}

Investment in A: $ 200,000
B: $ 200,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(91,0.8),(76,1.0)}

Investment in A: $ 100,000
B: $ 300,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(103,0.7),(85,1.0)}

Investment in A: $ 0
B: $ 400,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(94,0.9),(91,1.0)}*

<u>For the total investment of $ 300,000 in A and B:</u>

Investment in A: $ 300,000
B: $ 0: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(76,0.6),(63,1.0)}*

Investment in A: $ 200,000
B: $ 100,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(61,0.8),(54,1.0)}

Investment in A: $ 100,000
B: $ 200,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(74,0.8),(57,1.0)}

Investment in A: $ 0
B: $ 300,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(73,0.7),(60,1.0)}

<u>For the total investment of $ 200,000 in A and B:</u>

Investment in A: $ 200,000
B: $ 0: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(47,0.8),(44,1.0)}*

Investment in A: $ 100,000
B: $ 100,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(44,0.6),(35,1.0)}

Investment in A: $ 0
B: $ 200,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(44,0.8),(32,1.0)}

<u>For the total investment of $ 100,000 in A and B:</u>

Investment in A: $ 100,000
B: $ 0: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(30,0.6),(25,1.0)}*

Investment in A: $ 0
B: $ 100,000: $\tilde{C}_0^A \cup \tilde{C}_0^B$ = {(14,0.6),(10,1.0)}

Now we will devise all possible investments that encompass plans *A*, *B*, and *C*, and identify the most lucrative one.

Investment in A+B: $ 600,000
C: $ 0 : $\tilde{C}_1^{A+B} \cup \tilde{C}_1^C$ = {(96,0.7),(95,1.0)}

Investment in A+B: $ 500,000
C:$100,000: $\tilde{C}_1^{A+B} \cup \tilde{C}_1^C$ = {(141,0.9),(131,1.0)}

Investment in A+B: $ 400,000
C: $ 200,000: $\tilde{C}_1^{A+B} \cup \tilde{C}_1^C$ = {(141,0.9),(135,1.0)}*

Investment in A and B: $ 300,000
C: $ 300,000: $\tilde{C}_1^{A+B} \cup \tilde{C}_1^C$ = {(134,0.6),(111,1.0)}

Investment in A and B: $ 200,000
C: $ 400,000: $\tilde{C}_1^{A+B} \cup \tilde{C}_1^C$ = {(108,0.8),(100,1.0)}

Investment in A and B: $ 100,000

C: $ 500,000: $\tilde{C}_1^{A+B} \cup \tilde{C}_1^{C} = \{(111,0.9),(104,1.0)\}$

Investment in A and B: $ 0

C: $ 600,000: $\tilde{C}_1^{A+B} \cup \tilde{C}_1^{C} = \{(104,0.8),(102,1.0)\}$

Thus, we should invest in A: $ 0 and B: $ 400,000 and C: $ 200,000.

4. Conclusions

In the paper, we presented a fuzzy dynamic programming application for the selection of equal life and independent multi level investments. This method should be used when imprecise or fuzzy input data or parameters exist.

In multi level mathematical programming, input data or parameters are often imprecise or fuzzy in a wide variety of hierarchical optimization problems such as defence problems, transportation network designs, economical analysis, financial control, energy planning, government regulation, equipment scheduling, organizational management, quality assurance, conflict resolution and so on. Developing methodologies and new concepts for solving fuzzy and possibilistic multi-level programming problems is a practical and interesting direction for future studies.

For more about this subject, the readers should have [4], [5], [6].

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