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On the Fuzzy Control of Nonlinear Dynamic Systems with Inaccessible States

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Abstract

A systematic design method for PDC(Parallel Distributed Compensation)-type continuous time Takagi-Sugeno (T-S in short) fuzzy control systems which have inaccessible states is developed in this paper. Reduced-dimensional fuzzy state estimator is introduced from existing T-S fuzzy model using the PDC structure of Wang *et al.* [1]. LMI (Linear Matrix Inequalities) problems which represent the stability of the reduced-dimensional fuzzy state estimator are derived. Pole placement constraints idea for each rules is adopted to determine the estimator gains and they are also revealed as LMI problems. These LMI problems are combined with Joh *et al.*'s [7] [8] LMI problems for PDC-type continuous time T-S fuzzy controller design to yield a systematic design method for PDC-type continuous time T-S fuzzy control systems which have inaccessible states.

1 Introduction

Tanaka and Sugeno [2] proposed a theorem on the stability analysis of T-S fuzzy model. Wang *et al.* [1] proposed the so-called PDC as a design framework and also modified the Tanaka's stability theorem to include control. An important observation in the paper is that the stability problem is a standard feasibility problem with several LMIs when the feedback gains are predetermined and can be solved numerically using an algorithm named interior-point method. They are, however, NMIs (Nonlinear Matrix Inequalities) when the feedback gains are treated as unknowns. So, Wang *et al.*'s method can be considered as a stability checking method for pre-designed system and needs trial-and-error for control design.

Joh *et al.*'s [7] [8] converted the NMIs to LMIs for both of continuous and discrete T-S fuzzy controllers by applying the Schur complements [3, page 7] to the Wang *et al.*'s stability criterion and named it as stability LMIs. And they proposed a systematic design method based on the stability LMIs for T-S fuzzy controllers which guarantees global asymptotic stability and satisfies desired performance of the closed-loop system. It was accomplished by including LMIs about pole placement constraints to the stability LMIs and solving them numerically. The desired performance was represented as LMIs which is a region in the complex plane where the desired closed-loop poles lie inside of the desired region [4] [5].

Joh *et al.*'s [7] [8] work assumed that all the state variables are accessible. There are, however, many nonlinear systems which have inaccessible state variables. In this paper, Joh *et al.*'s [7] [8] method is expanded to include those cases. Reduced-dimensional fuzzy state estimator is proposed as r T-S type fuzzy rules using Wang *et al.*'s PDC structure to estimate the inaccessible states. Their stability LMIs are derived since the fuzzy state estimator may not be stable even though r individual estimators are stable. Pole placement constraint LMIs are derived for fuzzy state estimator to specify the desired responses. So, the estimator gains which guarantee stability and desired performances can be determined by solving the LMIs simultaneously. A systematic design method for PDC-type fuzzy controllers with inaccessible states is obtained by combining LMIs for fuzzy state estimator and LMIs for control.

2 Background Materials and Problem Formulations

2.1 T-S Fuzzy Model of Nonlinear Dynamic Systems and Its Stability

Takagi and Sugeno [6] proposed an effective way to represent a fuzzy model of nonlinear dynamic systems. It uses a linear input-output relation as its consequence of individual plant rule. A continuous time T-S fuzzy model is composed of r plant rules that can be represented as

$$\begin{array}{l}\text{if } x_1(t) \text{ is } M_1^i \text{ and } \cdots \text{ and } x_n(t) \text{ is } M_n^i \\ \text{then } \dot{x}(t) = A_i x(t) + B_i u(t), \quad i = 1, 2, \cdots, r\end{array} \tag{1}$$

where
x_j : j^{th} state (or linguistic) variable,
M_j : fuzzy term set of x_j,
M_j^i : a fuzzy term of M_j selected for plant rule i,
$x(t)$: state vector $\in R^n$, $u(t)$: input vector $\in R^m$,
$A_i \in R^{n\times n}$, $B_i \in R^{n\times m}$.

For any current state vector $x(t)$ and input vector $u(t)$, the T-S fuzzy model infers $\dot{x}(t)$ as the output of the fuzzy model as follows:

$$\dot{x}(t) = \frac{\sum_{i=1}^r w_i \left[A_i x(t) + B_i u(t)\right]}{\sum_{i=1}^r w_i} \tag{2}$$

where

$$w_i = \prod_{k=1}^{r} M_k^i(x_k(t)). \tag{3}$$

For a free system (i.e., $u(t) \equiv 0$), (2) can be written as

$$\dot{x}(t) = \frac{\sum_{i=1}^r w_i A_i x(t)}{\sum_{i=1}^r w_i}. \tag{4}$$

It is assumed, from now on, a proper continuous T-S fuzzy model is available.

Tanaka and Sugeno [2] suggested an important criterion for the stability of the T-S fuzzy model.

Theorem 1. [Stability Criterion for T-S Fuzzy Model] The equilibrium of the continuous-time T-S fuzzy model (4) (namely, $x = 0$) is globally asymptotically stable if there exists a common symmetric positive definite matrix

P such that

$$A_i^T P + PA_i < 0 \text{ for all } i = 1, 2, \cdots, r. \quad (5)$$

2.2 PDC-type T-S Fuzzy Control System and Its Stability

Wang *et al.* [1] proposed a framework which can be used as a guideline to design a T-S fuzzy controller using existing T-S fuzzy model. In this case, we can use a proper linear control method for each pair of control rule and plant rule. Wang *et al.* [1] named it PDC(Parallel Distributed compensation).

A PDC-type T-S fuzzy controller which uses full state feedback is composed of r control rules that can be represented as

$$\begin{array}{l}\text{if } x_1(t) \text{ is } M_1^i \text{ and } \cdots \text{ and } x_n(t) \text{ is } M_n^i \\ \text{then } u(t) = K_i x(t), \quad i = 1, 2, \cdots, r.\end{array} \quad (6)$$

For any current state vector $x(t)$, the T-S fuzzy controller infers $u(t)$ as the output of the fuzzy controller as follows:

$$u(t) = \frac{\sum_{j=1}^{r} w_j K_j x(t)}{\sum_{j=1}^{r} w_j}. \quad (7)$$

It has very important advantage because it makes easy (or manageable) to apply (7) to (2). Therefore the closed-loop behavior of the T-S fuzzy model (1) with the T-S fuzzy controller (6) using PDC can be obtained by substituting (7) into (2) as follows:

$$\dot{x}(t) = \frac{\sum_{i=1}^{r}\sum_{j=1}^{r} w_i w_j (A_i + B_i K_j) x(t)}{\sum_{i=1}^{r}\sum_{j=1}^{r} w_i w_j}. \quad (8)$$

The corresponding sufficient condition for the stability of (8) can be easily obtained.

Theorem 2. [Stability Condition for PDC-type T-S Fuzzy Control System] The equilibrium of the continuous-time PDC-type T-S fuzzy control system (namely, $x = 0$) is globally asymptotically stable if there exists a common symmetric positive definite matrix P such that

$$(A_i + B_i K_j)^T P + P(A_i + B_i K_j) < 0 \quad i, j = 1, 2, \cdots, r \quad (9)$$

It is suggested that P can be determined numerically by solving LMIs in (9) when K_i's, $i = 1, 2, \cdots, r$ are predetermined. It should be noted that (9) has r^2 LMIs. Wang *et al.* [1] rewrote (8) by grouping the same terms and the corresponding sufficient condition for stability is summarized in the Corollary 1.

Corollary 1. [Less Conservative Stability Condition for PDC-type T-S Fuzzy Control System] The equilibrium of the continuous-time PDC-type T-S fuzzy control system (namely, $x = 0$) is globally asymptotically stable if there exists a common symmetric positive definite matrix P such that

$$\begin{array}{r}(A_i + B_i K_i)^T P + P(A_i + B_i K_i) < 0, \; i = 1, 2, \cdots r \\ G_{ij}^T P + PG_{ij} < 0, \; i < j \leq r.\end{array} \quad (10)$$

where

$$G_{ij} = \frac{(A_i + B_i K_j) + (A_j + B_j K_i)}{2}, i < j \leq r. \quad (11)$$

The number of LMIs for (10) is $\frac{r(r+1)}{2}$. Therefore the number of LMIs to be solved is reduced greatly from r^2 of (9) to $\frac{r(r+1)}{2}$.

2.3 Stability LMIs for State Feedback Control and Pole Placement Design

It should be emphasized that the stability criterion (10) is LMIs if and only if the feedback gains K_i's are predetermined using proper design method in Wang *et al.* [1]. It is, however, necessary to treat K_i's and P as matrix variables when we try to find a systematic control design method which guarantees the stability criterion (10), i.e., they should be determined simultaneously. In that case, the stability criterion (10) is not LMIs. Any analytic or numerical method which solves such NMI problems does not exist yet. Joh *et al.* [7] [8] converted the NMIs to LMIs for continuous-time PDC-type T-S fuzzy control systems from (10) using the well-known Schur complements.

Theorem 3. [Stability LMIs for PDC-type T-S Fuzzy Control System] The equilibrium of the continuous-time PDC-type T-S fuzzy system (namely, $x = 0$) with unknown K_i's is globally asymptotically stable if there exists a common symmetric positive definite matrix $Q = P^{-1} > 0$ which satisfies

$$\begin{array}{r}QA_i^T + A_i Q + V_i^T B_i^T + B_i V_i < 0, \; i = 1, 2, \cdots, r \\ QA_i^T + A_i Q + QA_j^T + A_j Q + V_j^T B_i^T + B_i V_j \\ +V_i^T B_j^T + B_j V_i < 0, \; i < j \leq r\end{array} \quad (12)$$

where Q and $V_i = K_i Q$, $i = 1, 2, \cdots, r$ are new matrix variables of LMIs.

Stability LMIs play very important roles in the design of PDC-type T-S fuzzy control systems since they are able to treat the feedback gains as unknowns. It means they can be used as bases for invention of systematic design methods of PDC-type T-S fuzzy control systems.

Chilali and Gahinet [4] [5] proposed that a convex region which represents the desired closed-loop pole-placement constraints can be represented as LMIs.

Theorem 4. [LMIs for Pole Placement Constraints] The closed-loop poles lie in the LMI region

$$D = \{z \in C \mid f_D(z) := L + Mz + M^T \bar{z} < 0\} \quad (13)$$

if and only if there exists a symmetric positive definite matrix X_{pol} satisfying

$$\begin{array}{r}[\lambda_{ij} X_{pol} + \mu_{ij}(A + BK) X_{pol} + \mu_{ji} X_{pol} \\ \cdot (A + BK)^T]_{1 \leq i,j \leq m} < 0\end{array} \quad (14)$$

Here, z is a complex variable, A, B, and K are system, input, and feedback gain matrices of a linear system, respectively, and $L = L^T = [\lambda_{ij}]_{1 \leq i,j \leq m}$ and $M = [\mu_{ij}]_{1 \leq i,j \leq m}$ are known real matrices which can be determined by specifying desired closed-loop pole region in complex plane.

Joh *et al.* [7] [8] proposed a new design method which guarantees global asymptotic stability and satisfies desired performances by applying the Chilali and Gahinet's LMI regions [4] [5] to each local T-S fuzzy controller in order to specify the desired closed-loop performance. It means that we have r LMI regions corresponding to r local T-S fuzzy controller as follows:

$$[\lambda_{kl}Q + \mu_{kl}A_iQ + \mu_{kl}B_iV_i + \mu_{lk}QA_i^T + \mu_{lk}V_i^TB_i^T]_{1\le k,l\le m} < 0,\ i = 1,2,\cdots,r \tag{15}$$

where Q is used instead of X_{pol} since Q is symmetric positive definite and $K_iQ = V_i$. So, r LMIs in (15) can be used as the desired closed-loop pole placement constraints for T-S fuzzy control systems. Therefore, combination of the stability LMIs and (15) gives a new design method for T-S fuzzy controllers.

Theorem 5. [Pole Placement Design of PDC-type T-S Fuzzy Control System] A continuous-time PDC-type T-S fuzzy controller which guarantees global asymptotic stability and satisfies desired performance by placing closed-loop poles for each local model within the desired region can be designed by solving

$$\begin{aligned}
&QA_i^T + A_iQ + V_i^TB_i^T + B_iV_i < 0,\ i = 1,2,\cdots,r\\
&QA_i^T + A_iQ + QA_j^T + A_jQ + V_j^TB_i^T + B_iV_j\\
&\qquad + V_i^TB_j^T + B_jV_i < 0,\ i < j \le r\\
&[\lambda_{kl}Q + \mu_{kl}A_iQ + \mu_{kl}B_iV_i + \mu_{lk}QA_i^T +\\
&\qquad \mu_{lk}V_i^TB_i^T]_{1\le k,l\le m} < 0,\ i = 1,2,\cdots,r\\
&Q > \alpha I,\ \alpha = \text{positive constant}
\end{aligned} \tag{16}$$

We can obtain Q and V_i, $i = 1,2,\cdots,r$, by solving (16). And then the common symmetric positive definite matrix P and feedback gains K_i, $i = 1,2,\cdots,r$, can be determined as follows:

$$P = Q^{-1},\ \ K_i = V_iQ^{-1} = V_iP,\ i = 1,2,\cdots,r. \tag{17}$$

2.4 Motivation and Problem Formulations

It is assumed in the previous subsections that all the state variables can be measured. There are, however, many cases where all or some of the state variables can not be measured. We may be able to overcome this problem by introducing fuzzy state estimator under the assumption of observerbility of the consequence part of (1). If the well-known separation property holds, the gains of the feedback controller and fuzzy state estimator can be designed separately and systematically using pole placement constraints in the subsection 2.3. This is the motivation of the research in this paper. So, the problems in this paper can be defined by proposing the fuzzy state estimator. The reduced-dimensional fuzzy state estimator is addressed in this paper.

From now, the state variable $x_j(t)$ of the antecedent part of (1) is substituted by $\hat{x}_j(t)$, the estimate of $x_j(t)$, since the inaccessible state variables should be estimated for the state feedback control.

Proposition 1. [Reduced-dimensional Fuzzy State Estimator] We assume that some of the state variables are not accessible. Also, for simplicity, we assume that the plant dynamic models of consequence part of (1) have observable companion form as follows

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1^i \\ B_2^i \end{bmatrix}u(t)\\
y(t) &= x_1(t)
\end{aligned} \tag{18}$$

where

$y(t)$: output state variables,
$x_1(t)$: accessible state variables,
$x_2(t)$: inaccessible state variables.

If the dynamic equation of consequence part in (1) is observable, a $(n - q)$-dimensional fuzzy state estimator can be constructed using the similar structure to PDC to estimate inaccessible states for each rules. Note that the rank of the output matrix is q. The i^{th} rule of the reduced-dimensional fuzzy state estimator can be written by as follows

$$\begin{aligned}
&\text{if } \hat{x}_1(t) \text{ is } M_1^i \text{ and } \cdots \text{ and } \hat{x}_n(t) \text{ is } M_n^i\\
&\text{then } \dot{z}(t) = F_iz(t) + G_iy(t) + H_iu(t),\ i = 1,2,\cdots,r
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
&\hat{x}_2(t) = z(t) - L_iy(t),\\
&F_i = A_{22}^i + L_iA_{12}^i,\\
&G_i = -(A_{22}^i + L_iA_{12}^i)L_i + (A_{21}^i + L_iA_{11}^i),\\
&H_i = B_2^i + L_iB_1^i.
\end{aligned}$$

L_i : $(n - q) \times q$ real constant observer gain matrix,

Therefore, (19) infers the output of the reduced-dimensional fuzzy state estimator $\dot{z}(t)$ as follows:

$$\dot{z}(t) = \frac{\sum_{i=1}^r w_i\,[F_iz(t) + G_iy(t) + H_iu(t)]}{\sum_{i=1}^r w_i}. \tag{20}$$

where

$$w_i = \prod_{k=1}^r M_k^i(\hat{x}_k(t)). \tag{21}$$

A PDC-type T-S fuzzy controller which uses feedback of estimated states is composed of r control rules that can be represented as

$$\begin{aligned}
&\text{if } \hat{x}_1(t) \text{ is } M_1^i \text{ and } \cdots \text{ and } \hat{x}_n(t) \text{ is } M_n^i\\
&\text{then } u(t) = K_i\hat{x}(t),\ \ i = 1,2,\cdots,r.
\end{aligned} \tag{22}$$

3 Stability LMIs with Fuzzy State Estimators

Theorem 3 which defines stability LMIs play very important roles for design of T-S fuzzy controllers with state feedback. Similar expressions are necessary for the case of using the fuzzy estimators for the design of PDC-type T-S fuzzy controllers with inaccessible states.

Stability of the T-S fuzzy control system represented as (1), (19), and (22) can be investigated using the following composite dynamic equation.

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \bar{A}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{23}$$

where

$$\bar{A} = \begin{bmatrix} \frac{\sum_{i=1}^{r}\sum_{j=1}^{r} w_i w_j (A_i + B_i K_j)}{W} & \frac{\sum_{i=1}^{r}\sum_{j=1}^{r} w_i w_j B_i K_j Q_{2i}}{W} \\ 0 & \frac{\sum_{i=1}^{r}\sum_{j=1}^{r} w_i w_j F_i}{W} \end{bmatrix} \tag{24}$$

$$W = \sum_{i=1}^{r}\sum_{j=1}^{r} w_i w_j \tag{25}$$

, $e(t) = z(t) - T_i x(t)$, $T_i = [0 \;\; I_{n-q}]$, and $\begin{bmatrix} C \\ T_i \end{bmatrix}^{-1} = [\, Q_{1i} \;\; Q_{2i} \,]$.

The corresponding sufficient stability condition for (23) can be obtained as follows.

Theorem 6. [Stability LMIs for PDC-type T-S Fuzzy Control System Using Reduced-dimensional Fuzzy State Estimator] The equilibrium of (23) (namely, $x = 0$ and $e = 0$) is globally asymptotically stable if there exists a common symmetric positive definite matrix $\bar{P}$ such that

$$\bar{A}^T \bar{P} + \bar{P}\bar{A} < 0. \tag{26}$$

Eq. (26) is a NMI since K_i's and L_i's are unknowns. It is equivalent to finding the $P > 0$ and $Q > 0$ of the following two LMIs

$$\begin{aligned} & A_i Q + Q A_i^T + B_i V_j + V_j^T B_i^T < 0, \; i,j = 1,2,\cdots,r \\ & P A_{22}^i + {A_{22}^i}^T P + W_i A_{12}^i + {A_{12}^i}^T W_i^T < 0, \\ & \qquad\qquad\qquad\qquad i = 1,2,\cdots,r \end{aligned} \tag{27}$$

when $\bar{P}$ is restricted to

$$\bar{P} = \begin{bmatrix} \lambda Q^{-1} & 0 \\ 0 & P \end{bmatrix} \tag{28}$$

where λ is a positive scalar. It should be noted that Q, P, V_j's, and W_i's are matrix variables and

$$V_i = K_i Q \;\text{ and }\; W_i = P L_i \tag{29}$$

where K_i's and L_i's are feedback gains and estimator gains, respectively.

The Theorem 6 can be made less conservative by using the similar concept in the Corollary 1 as follows.

Theorem 7.[Less Conservative Stability LMIs for PDC-type T-S Fuzzy Control System Using Reduced-dimensional Fuzzy State Estimator] The equilibrium of (23) (namely, $x = 0$ and $e = 0$) is globally asymptotically stable if there exists a common symmetric positive definite matrix P and Q such that

$$\begin{aligned} & A_i Q + Q A_i^T + B_i V_i + V_i^T B_i^T < 0, \;\; i = 1,2,\cdots,r \\ & A_i Q + Q A_i^T + A_j Q + Q A_j^T + B_i V_j + V_j^T B_i^T \\ & \qquad + B_j V_i + V_i^T B_j^T < 0, \;\; i < j \le r \\ & P A_{22}^i + {A_{22}^i}^T P + W_i A_{12}^i + {A_{12}^i}^T W_i^T < 0, \\ & \qquad\qquad\qquad\qquad i = 1,2,\cdots,r. \end{aligned} \tag{30}$$

Remark 1. It should be noted that Theorem 7 reveals the well-known separation property in the linear systems theory since $V_i = K_i Q$ and $W_i = P L_i$ are not coupled and can be designed separately.

4 Design of T-S Fuzzy Controllers using Fuzzy State Estimator

Joh *et al.* [7] [8] proposed that the Chilali and Gahinet's LMI regions [4] [5] for each local T-S fuzzy controller can be combined with the stability LMIs in order to specify the desired transient response as shown in the Section 2. The same idea can be applied to the design of T-S fuzzy controllers which uses fuzzy state estimators. It should be emphasized that we can separate the design of feedback gains and estimator gains since the well-known separation property holds as shown in the Section 3. Therefore, we can make a systematic design method for PDC-type continuous time T-S fuzzy control system which uses fuzzy state estimator by adding the pole placement design for the fuzzy state estimator. In here, LMI region corresponding to each local T-S fuzzy estimator in (19) is given as the following theorem.

Theorem 8. [Pole Placement Constraint LMIs for Fuzzy Estimator] The poles of the reduced-dimensional fuzzy observer lie in the LMI region

$$D = \{ z \in C \mid f_D(z) := L^* + M^* z + {M^*}^T \bar{z} < 0 \} \tag{31}$$

if and only if there exists a symmetric positive definite matrix P satisfying

$$\begin{aligned} [\lambda_{kl}^* P + \mu_{kl}^* P A_{22}^i + \mu_{kl}^* W_i A_{12}^i + \mu_{lk}^* {A_{22}^i}^T P \\ + \mu_{lk}^* {A_{12}^i}^T W_i^T]_{1 \le k,l \le m} < 0 \end{aligned} \tag{32}$$

Now, combinations of the stability LMIs and pole placement constraint LMIs yield new design methods for PDC-type T-S fuzzy control systems with fuzzy state estimators. They are summarized in the following theorem.

Theorem 9. [Pole Placement Design of PDC-type T-S Fuzzy Control System using Reduced-dimensional Fuzzy State Estimator] Consider a dynamic system some of whose state variables are not accessible. A continuous-time PDC-type T-S fuzzy controller which guarantees global asymptotic stability and satisfies desired performance can be designed by solving

$$\begin{aligned} & A_i Q + Q A_i^T + B_i V_i + V_i^T B_i^T < 0, \; i = 1,2,\cdots,r \\ & A_i Q + Q A_i^T + A_j Q + Q A_j^T + B_i V_j + V_j^T B_i^T + B_j V_i \\ & \qquad + V_i^T B_j^T < 0, \; i < j \le r \\ & [\lambda_{kl} Q + \mu_{kl} A_i Q + \mu_{kl} B_i V_i + \mu_{lk} Q A_i^T \\ & \qquad + \mu_{lk} V_i^T B_i^T]_{1 \le k,l \le m} < 0, \; i = 1,2,\cdots,r \\ & Q > \alpha I, \quad \alpha = \text{positive constant} \end{aligned} \tag{33}$$

$$\begin{aligned} & P A_{22}^i + {A_{22}^i}^T P + W_i A_{12}^i + {A_{12}^i}^T W_i^T < 0, \;\; i = 1,2,\cdots,r \\ & [\lambda_{kl}^* P + \mu_{kl}^* P {A_{22}}^i + \mu_{kl}^* W_i {A_{12}}^i + \mu_{lk}^* {A_{22}}^{i^T} P \\ & \qquad + \mu_{lk}^* {A_{12}^i}^T W_i^T]_{1 \le k,l \le m} < 0, \; i = 1,2,\cdots,r \\ & P > \gamma I, \quad \gamma = \text{positive constant} \end{aligned} \tag{34}$$

where (33) defines the stability and the desired performance of the whole system and (34) defines the stability and the desired performance of the reduced-dimensional fuzzy state estimator.

5 An Simulated Example

The proposed design method is verified by designing a controller for an inverted pendulum with a cart which is adopted from Wang *et al.* [1]. The equation of motion for the pendulum are

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = \frac{g\sin(x_1(t)) - amlx_2{}^2(t)\sin(2x_1(t))/2 - a\cos(x_1(t))u(t)}{4l/3 - aml\cos^2(x_1(t))}. \quad (35)$$

where $x_1(t)$ is the angle(in radians) of the pendulum from the vertical, $x_2(t)$ is the angular velocity, and $u(t)$ is the control force(in Newton) applied to the cart. The other parameters are as follows:

- g : the gravity constant$(9.8m/s^2)$,
- m : mass of the pendulum$(2.0Kg)$,
- M : mass of the cart$(8.0Kg)$,
- $2l$: length of the pendulum$(1.0m)$,
- $a = \frac{1}{m+M}$.

The T-S fuzzy model in Wang *et al.* [1] is adopted in this paper. It is composed of two plant rules.

$$\begin{aligned}&\text{if } \hat{x}_1 \text{ is about } 0 \text{ then } \dot{x} = A_1x + B_1u\\ &\text{if } \hat{x}_1 \text{ is about } \pm\tfrac{\pi}{2}\ \left(|\hat{x}_1| < \tfrac{\pi}{2}\right) \text{ then } \dot{x} = A_2x + B_2u\end{aligned} \quad (36)$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ \frac{g}{4l/3 - aml} & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -\frac{a}{4l/3 - aml} \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 1 \\ \frac{2g}{\pi(4l/3 - aml\beta^2)} & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -\frac{a\beta}{4l/3 - aml\beta^2} \end{bmatrix}$$

and where $\beta = \cos(88^\circ)$. Refer to Wang *et al.* [1] for detailed description. Membership functions of "about 0" and "about $\pm\frac{\pi}{2}$" are shown in Fig. 1.

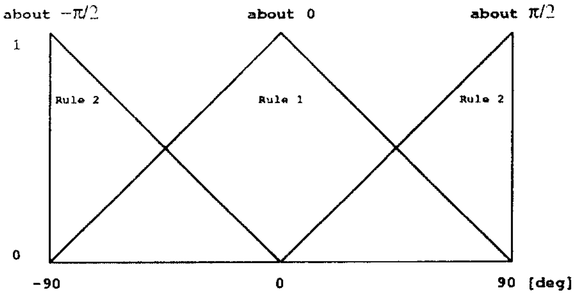

Figure 1: Membership functions of "about 0" and "about $\pm\frac{\pi}{2}$"

Let's assume that state variable $x_2(t)$ is inaccessible. (i.e., system output y is Cx and $C = [1 \;\; 0]$.) We can construct reduced-dimensional fuzzy state estimator to estimate $x_2(t)$ since $\{A_i, C\}$ in (36) is observable for all i. Furthermore, we can see that the plant dynamic models of consequence part of (36) are observable companion form. Therefore, the reduced-dimensional fuzzy state estimator for this example can be written as follows

$$\begin{aligned}&\text{if } \hat{x}_1 \text{ is about } 0 \text{ then } \dot{z} = F_1z + G_1y + H_1u\\ &\text{if } \hat{x}_1 \text{ is about } \pm\tfrac{\pi}{2}\ \left(|\hat{x}_1| < \tfrac{\pi}{2}\right)\\ &\qquad \text{then } \dot{z} = F_2z + G_2y + H_2u\end{aligned} \quad (37)$$

where

$$G_1 = \left[-L_1^2 + \frac{g}{4l/3 - aml}\right] \quad H_1 = \left[-\frac{a}{4l/3 - aml}\right]$$
$$G_2 = \left[-L_2^2 + \frac{2g}{\pi(4l/3 - aml\beta^2)}\right] \quad H_2 = \left[-\frac{a\beta}{4l/3 - aml\beta^2}\right].$$

,and $F_{1,2} = [L_{1,2}]$. The corresponding PDC-type T-S fuzzy controller can be represented as

$$\begin{aligned}&\text{if } \hat{x}_1 \text{ is about } 0 \text{ then } u = K_1\hat{x}\\ &\text{if } \hat{x}_1 \text{ is about } \pm\tfrac{\pi}{2}\ \left(|x_1| < \tfrac{\pi}{2}\right)\ then\ u = K_2\hat{x}\end{aligned} \quad (38)$$

and the resulting output of the controller is

$$u = \omega_1 K_1\hat{x} + \omega_2 K_2\hat{x} \quad (39)$$

since $\omega_1 + \omega_2 = 1$ from Fig. 1. Here, ω_1 and ω_2 are membership grades of antecedent parts of control rules 1 and 2 respectively.

The design purpose of this example is to place the closed-loop poles of each local model within the desired region as shown in Fig. 2 as shaded polygon. It corresponds to restrict damping and response time within certain range. Since the plant is the 2nd-order system, the response should be

$$\zeta > 0.995 \text{ and } T_s < 2.67\,(sec). \quad (40)$$

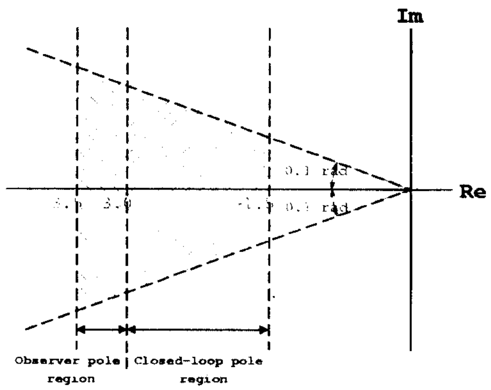

Figure 2: Desired Pole Placement Constraint Region

Therefore, the LMI region for the closed-loop system is defined by L and M matrix as

$$L = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & -m_{34} & m_{33} \end{bmatrix} \quad (41)$$

where $m_{33} = 0.0998$ and $m_{34} = -0.995$. We have to design the fuzzy state estimator to converge much faster than the closed-loop system. The desired LMI region for the estimator poles are shown in the Figure 2. The corresponding L^* and M^* matrices are

$$L^* = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; M^* = M. \tag{42}$$

Now, we have to solve 11 LMIs in (33) and (34) since $r = 2$. We can obtain the solution as follows

$$\begin{aligned} Q &= \begin{bmatrix} 2.179 \times 10^3 & -6.153 \times 10^3 \\ -6.153 \times 10^3 & 1.795 \times 10^4 \end{bmatrix} \\ K_1 &= \begin{bmatrix} 126.07 & 26.3 \end{bmatrix}, \; K_2 = \begin{bmatrix} 2722.3 & 883.4 \end{bmatrix} \\ P &= 3.532, \; L_{1,2} = 3.407. \end{aligned} \tag{43}$$

Since the eigen-values of matrix Q are 63 and 20066, Q and P are obviously symmetric positive definite matrices. Therefore, the PDC-type T-S fuzzy controller (38) using the reduced-dimensional fuzzy state estimator (37) is globally asymptotically stable. It verifies the stability of the proposed design method.

The performance of the proposed controller is checked by simulation. The simulation is performed with various initial conditions to see the performance of controlling nonlinear system. Figure 3 show the resulting response of the system for various initial conditions (i.e., $x_1 = 5, 25, 45, 65, 85$(deg.), and $x_2 = 0$). Also, performance of the proposed controller is compared to linear controller with reduced-dimensional estimator. The feedback gain and reduced-dimensional estimator gain selected for linear controller are K_1 and L_1 respectively. The solid lines indicate responses with the proposed fuzzy controller and the dotted lines show those with the linear controller in Fig. 3. In Fig. 3, we show that the performance specifications (40) are satisfied. Therefore, the performance of the proposed method is verified.

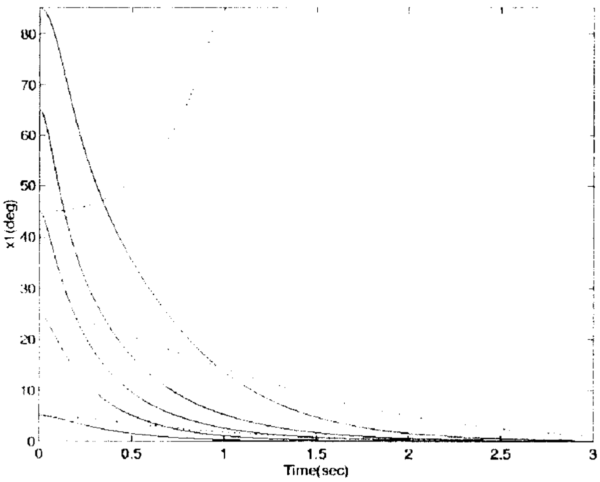

Figure 3: Position responses using linear and proposed fuzzy control

6 Concluding Remarks

A new design method for the continuous time T-S fuzzy controller with fuzzy estimators is proposed. The method uses LMI approach to find the common symmetric positive definite matrices Q and P and feedback gains , and estimator gains numerically. As the results of the stability analysis for the composite system, we can see that the separation principle between controller and estimator is satisfied. Therefore, it is possible to solve each stability LMIs for the closed-loop system and estimator. By solving stability LMIs and pole placement constraint LMIs for each closed-loop fuzzy system and fuzzy estimator simultaneously, the feedback gains and estimator gains which guarantee global stability and satisfy desired performance can be determined.

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References

[1] H. O. Wang, K. Tanaka, and M. Griffin, "An Analytical Framework of Fuzzy Modeling and Control of Nonlinear Systems: Stability and Design Issues," *Proc. American Control Conference*, Seattle, Washington, pp. 2272-2276, 1995.

[2] K. Tanaka and M. Sugeno, "Stability Analysis and Design of Fuzzy Control Systems," *Fuzzy Sets and Systems*, Vol. 45, pp. 135-156, 1992.

[3] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, "Linear Matrix Inequalities in System and Control Theory," *SIAM Studies in Applied Mathematics*, 1994.

[4] M. Chilali and P. Gahinet, "H_∞ Design with Pole Placement Constraints: an LMI Approach," *IEEE Trans. on Automatic Control*, Vol. 41, No. 3, March 1996.

[5] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, "LMI Control Toolbox: For Use with MATLAB," *The MATH WORKS Inc.*, 1995.

[6] T. Takagi and M. Sugeno, "Fuzzy Identification of Systems and Its Applications to Modeling and Control," *IEEE Trans. Systems, Man, and Cybernetics*, Vol. SMC-15, No.1, pp. 116-132, 1985.

[7] J. Joh, R. Langari, and W. J. Chung, "A New Design Method for Continuous Takagi-Sugeno Fuzzy Controller with Pole Placement Constraints: An LMI Approach," *Proc. 1997 IEEE Int. Conf. on Systems, Man, and Cybernetics*, Orlando, Florida, pp. 2969-2974, 1997.

[8] J. Joh, S. K. Hong, Y. Nam, and W. J. Chung, "On the Systematic Design of Takagi-Sugeno Fuzzy Control Systems," *Proc. Int. Symp. on Engineering of Intelligent Systems/ EIS'98*, Tenerife, Spain, Feb. 113-119, 1998.