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γ-FUZZY FILTER AND LIMIT STRUCTURE

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Abstract

We introduce the notion of γ-fuzzy filter and γ-limit structure to L-fuzzy points. We show that the category γLim of γ-limit spaces is a cartesian closed topological construct containing the category LFTop* of stratified L-fuzzy topological spaces as a bireflective subcategory.

Keywords : γ-fuzzy filter, (stratified) L-fuzzy topology, γ-limit structure.

1. Introduction

In general and fuzzy topological spaces the notion of limit structures with respect to points and fuzzy points was defined as to generalize the notion of neighborhood systems which characterizes topologies. So it provided a good tool of interpreting the topological structures. Filters and prefilters on a set X, the subjects of convergence, were defined as a subset of L^X, where $L = \{0,1\}$ and $L = [0,1]$, respectively and in [1], P. Eklund and W. Gähler made some generalization of filter by defining L-fuzzy filter as a map from L^X to L, where L is a completely distributive complete lattice. But it seems to be impossible for L-fuzzy filter to include prefilter which generalizes usual filter. In fact, a prefilter on a set X is a map from $[0,1]^X$ to $\{0,1\}$.

In this paper, we generalize L-fuzzy filter by introducing two completely distributive chains L, L' and a map $\gamma : L \to L'$ satisfying some conditions. We define the notion of γ-fuzzy filter on a set X by a map from L^X to L'. Then it can include not only L-fuzzy filter with completely distributive chain L but also prefilter. We introduce the notion of γ-limit structure of γ-fuzzy filters to L-fuzzy points. Then this structure turns out to be a generalization of both limitierung and fuzzy limitierung in general and fuzzy topological spaces, respectively. And from the categorical point of view the category γ**Lim** with objects γ-limit spaces and morphisms γ-continuous maps is shown to be cartesian closed topological category containing the category L**FTop*** of stratified L-fuzzy topological spaces and continuous maps as a bireflective subcategory. It is a generalized result of those in general and fuzzy topological spaces since we use Lowen's definition of fuzzy topology in limit structure theory.

2. Preliminaries

Let X be a set. Let $\mathcal{F}(X)$ be the collection of all prefilters on X and $\mathcal{X}$ be the set of all fuzzy points in X.

A *fuzzy limitierung* [6] Δ is a map from $\mathcal{X}$ into $\mathcal{P}(\mathcal{F}(X))$, subject to the following axioms: for each $x_\lambda \in \mathcal{X}$,

(L0) $\mathcal{F} \in \Delta(x_\lambda)$ implies $\overline{\alpha} \in \mathcal{F}$ for all $\alpha > 1 - \lambda$, where $\overline{\alpha}$ is a constant map on X with value $\alpha \in I$,

(L1) $\dot{x_\lambda} = \{A \in I^X : \mu_A(x) > 1 - \lambda\} \in \Delta(x_\lambda)$,

(L2) if $\mathcal{F} \in \Delta(x_\lambda)$ and $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{G} \in \Delta(x_\lambda)$,

(L3) if $\mathcal{F}, \mathcal{G} \in \Delta(x_\lambda)$, then $\mathcal{F} \cap \mathcal{G} \in \Delta(x_\lambda)$.

The pair (X, Δ) is called a *fuzzy limit space*. Given fuzzy limit spaces (X, Δ) and (Y, Δ') the map $f : X \to Y$ is said to be *fuzzy continuous* if for any fuzzy point x_λ in X and $\mathcal{F} \in \Delta(x_\lambda)$, the prefilter $f(\mathcal{F}) = \{B \in I^Y : f(A) \leq B \text{ for some } A \in \mathcal{F}\} \in \Delta'(f(x)_\lambda)$.

Let **FLim** be the category of all fuzzy limit spaces and all fuzzy continuous maps between them.

Theorem 2.1 **FLim** *is a cartesian closed topological category.*

Theorem 2.2 **FTop** *is a bicoreflective subcategory of* **FLim**.

If we substitute $[0,1]$ by $\{0,1\}$, the two point chain, the notions of fuzzy topology and fuzzy limitierung are equivalent those of topology and limitierung, respectively and we have the analogous result.

Theorem 2.3 Lim *is a cartesian closed topological category.*

Theorem 2.4 Top *is a bicoreflective subcategory of* **Lim**.

For any completely distributive complete lattice L with different least and last element 0_L and 1_L and for any set X, an *L-fuzzy filter* [3] on X is a map $\mathcal{M} : L^X \to L$ such that the following conditions are fulfilled:

(F1) $\mathcal{M}(\overline{\alpha}) \leq \alpha$ for all $\alpha \in L$ and $\mathcal{M}(\overline{1}) = 1$,

(F2) $\mathcal{M}(f \wedge g) = \mathcal{M}(f) \wedge \mathcal{M}(g)$ for all $f, g \in L^X$.

3. L-fuzzy topological space and γ-fuzzy filter

Let L be a completely distributive chain with different least and last elements denoted by 0_L and 1_L. For any set X an *L-fuzzy topology* t on X is a non-empty subset of L^X which contains 0_L, 1_L and is closed under arbitrary supremum and finite infimum. A pair (X, t) is called an *L-fuzzy topological space* (for short, *L-fts*) and the elements of t are called *open L-fuzzy subsets* of (X, t). If s and t are L-fuzzy topologies on X, s is said to be *finer* than t and t is said to be *coarser* than s provided $t \subseteq s$ holds. An L-fuzzy topology t is said to be *stratified* if $\overline{\alpha} \in t$ for all $\alpha \in L$.

In an L-fts (X, t), define int : $L^X \to L^X$ by

$$\text{int} f = \bigvee_{\substack{g \in t \\ g \leq f}} g.$$

Then int f is the greatest open L-fuzzy subset less than or equal to f and the map int satisfies the following properties.

(I1) int $\overline{1_L} = \overline{1_L}$.

(I2) int $f \leq f$ and int(int f) = int f for all $f \in L^X$.

(I3) int $(f \wedge g)$ =int $f \wedge$ int g for all $f, g \in L^X$.

Conversely, if for any $f \in L^X$ there corresponds an L-fuzzy set int f such that the conditions (I1), (I2) and (I3) are satisfied, then there exists a unique L-fuzzy topology $t = \{f \in L^X : \text{int} f = f\}$ on X with interior operator int.

Let L' be another completely distributive chain with different least and last element $0_{L'}$ and $1_{L'}$ and $\gamma : L \to L'$ be a surjective map satisfying the following conditions:
(i) $\gamma(1_L) = 1_{L'}$ and $\gamma(\alpha) = 0_{L'}$ iff $\alpha = 0_L$,
(ii) γ preserves the arbitrary supremum.

From now on, let completely distributive chains L, L' and the map $\gamma : L \to L'$ be fixed.

Definition 3.1 Let X be a non-empty set. A mapping $\mathcal{M} : L^X \to L'$ is called a *γ-fuzzy filter* if it satisfies the following conditions:

(γF1) $\mathcal{M}(\overline{\alpha}) \leq \gamma(\alpha)$ for all $\alpha \in L$ and $\mathcal{M}(\overline{1_L}) = 1_{L'}$,

(γF2) $\mathcal{M}(f \wedge g) = \mathcal{M}(f) \wedge \mathcal{M}(g)$ for all $f, g \in L^X$.

Clearly, $\mathcal{M}(\overline{0_L}) = 0_{L'}$ and $\mathcal{M}$ preserves the order of L^X.

Example 3.2 (i) If $L = L' = \{0, 1\}$, then $\gamma : L \to L' = id_X$ and the γ-fuzzy filter is an usual filter.
(ii) If $L = [0, 1]$ and $L' = \{0, 1\}$, then $\gamma : L \to L'$ is $\alpha \longmapsto 0$ if $\alpha = 0$ and $\alpha \longmapsto 1$ if $\alpha > 0$ and the γ-fuzzy filter is a prefilter.
(iii) If $L = L'$ is an arbitrary completely distributive chain and $\gamma : L \to L' = id_L$, then the γ-fuzzy filter is an L-fuzzy filter [3].

For any set X, let $\mathcal{F}_\gamma(X)$ be the set of all γ-fuzzy filters on X and for $\mathcal{M}, \mathcal{N} \in \mathcal{F}_\gamma(X)$, we say that $\mathcal{M}$ is *finer* than $\mathcal{N}$ and $\mathcal{N}$ is *coarser* than $\mathcal{M}$ if $\mathcal{M}(f) \geq \mathcal{N}(f)$ for all $f \in L^X$ and we denote this by $\mathcal{M} \geq \mathcal{N}$. With this order, $\mathcal{F}_\gamma(X)$ is a partially ordered set.

For any sets X and Y and a map $f : X \to Y$, the map $\mathcal{F}_\gamma f(\mathcal{M}) : L^Y \to L'$ defined by $\mathcal{F}_\gamma f(\mathcal{M})(g) = \mathcal{M}(g \circ f)$ for all $g \in L^Y$ is a γ-fuzzy filter on Y and we write $\mathcal{F}_\gamma f(\mathcal{M})$ simply by $f(\mathcal{M})$.

Notice that for any completely distributive chain $(L, \leq)$, the associated pair $(L, \dot{\leq})$, where $x \dot{\leq} y$ iff $x \geq y$ for all $x, y \in L$ is also a completely distributive chain.

In the sequel we assume that L satisfies the condition : there exists a unique order-preserving one-to-one correspondence from $(L, \leq)$ to $(L, \dot{\leq})$.
We denote the image of λ by this map by λ^*.

For $\gamma : L \to L'$ and $\lambda > 0_L$, define a map $\gamma_\lambda : L \to L'$ by

$$\gamma_\lambda(\alpha) = \begin{cases} 0_{L'} & \text{if } \alpha \leq \lambda^* \\ \gamma(\alpha) & \text{if } \alpha > \lambda^*. \end{cases}$$

Proposition 3.3 *γ_λ preserves finite infimum and arbitrary supremum.*

Proof. It can be easily checked.

For any set X, $x \in X$ and $\lambda > 0_L$, an L-fuzzy point x_λ in X is an L-fuzzy set on X defined by

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x \\ 0_L & \text{otherwise.} \end{cases}$$

Proposition 3.4 *For any L-fuzzy point x_λ,*

$$\begin{array}{rcl} \dot{x_\lambda} : L^X & \longrightarrow & L' \\ f & \longmapsto & \dot{x_\lambda}(f) = \gamma_\lambda(f(x)) \end{array}$$

is a γ-fuzzy filter.

Proof. (γF1) is immediate from the definition of $\dot{x}_\lambda$ and (γF2) follows from Proposition 3.3.

So there exists a one-to-one correspondence between L-fuzzy topologies and interior operators.

Proposition 3.5 *Let (X,t) be an L-fts. Then for any L-fuzzy point x_λ in X, the map*

$$\begin{array}{rcl} \mathcal{N}_t(x_\lambda) : L^X & \longrightarrow & L' \\ f & \longmapsto & \mathcal{N}_t(x_\lambda)(f) = \gamma_\lambda(\operatorname{int} f(x)) \end{array}$$

is a γ-fuzzy filter.

Proof. It is immediate from (I2), (I3) and Proposition 3.3.

We call the map $\mathcal{N}_t(x_\lambda)$ by *γ-fuzzy neighborhood filter* of x_λ in an L-fts (X,t).

Remark 3.6 *Let (X,t) be an L-fts. Then for any L-fuzzy point x_λ in X and $f \in L^X$,*

$$\begin{aligned} \mathcal{N}_t(x_\lambda)(f) &= \gamma_\lambda(\operatorname{int} f(x)) \\ &= \gamma_\lambda\Big(\bigvee_{\substack{g\in t\\ g\le f}} g(x)\Big) = \bigvee_{\substack{g\in t\\ g\le f}} \gamma_\lambda(g(x)). \end{aligned}$$

Proposition 3.7 *Let (X,t) be an L-fts. Then an L-fuzzy set $f \in t$ iff for any L-fuzzy point x_λ in X and $\beta \in L'$, $\gamma_\lambda(f(x)) > \beta$ implies $\mathcal{N}_t(x_\lambda)(f) > \beta$.*

Proof. The only if part is immediate from the fact that $\operatorname{int} f = f$ for all $f \in t$. To show the converse, we note that by the assumption and Remark 3.6, it holds for any L-fuzzy point x_λ in X and $\beta \in L'$ if $\gamma_\lambda(f(x)) > \beta$ then there exists $g \in t$ such that $g \le f$ and $\gamma_\lambda(g(x)) > \beta$. Let $x \in X$ such that $f(x) > 0_L$ and $\lambda > f(x)^*$. Then $\gamma_\lambda(f(x)) > 0_{L'}$ and hence there exists $g_{x_\lambda} \in t$ such that $g_{x_\lambda} \le f$ and $\gamma_\lambda(g_{x_\lambda}(x)) > 0_{L'}$. Let $g_x = \vee_{\lambda > f(x)^*} g_{x_\lambda}$, then $g_x \in t$ and $g_x \le f$. Furthermore, $g_x(x) = f(x)$ since $\gamma_\lambda(g_x(x)) > 0_{L'}$ for any $\lambda > f(x)^*$. Let

$$h = \bigvee_{\substack{x\in X\\ f(x)>0_L}} g_x.$$

Then $f = h$ and we have the result.

Proposition 3.8 *In an L-fts (X,t), $\mathcal{N}_t(x_\lambda) \le \dot{x}_\lambda$ for all L-fuzzy point x_λ in X and the equality holds iff $f \in t$.*

Proof. It is immediate from Proposition 3.7.

Definition 3.9 Given L-fts's (X,t) and (Y,t'), the map $f : X \to Y$ is said to be *continuous* at an L-fuzzy point x_λ in X if $\mathcal{N}_{t'}(f(x)_\lambda) \le f(\mathcal{N}_t(x_\lambda))$, i.e $\mathcal{N}_{t'}(f(x)_\lambda)(g) \le \mathcal{N}_t(x_\lambda)(g \circ f)$ for all $g \in L^Y$ and f is said to be *continuous* if f is continuous at each L-fuzzy point in X.

Let L**FTop** be the category of all L-fts's and all continuous maps and let L**FTop*** be the full subcategory of L**FTop** consisting of all stratified L-fts's.

Proposition 3.10 *For any L-fts's (X,t) and (Y,t') a map $f : X \to Y$ is continuous iff $g \in t'$ implies $g \circ f \in t$.*

Proof. Take any L-fuzzy point x_λ in X and $\beta \in L'$ such that $\gamma_\lambda((g\circ f)(x)) > \beta$. Then $\gamma_\lambda(g(f(x))) > \beta$ and hence $\mathcal{N}_{t'}(f(x)_\lambda)(g) > \beta$. Since $\mathcal{N}_t(x_\lambda)(g\circ f) \ge \mathcal{N}_{t'}(f(x)_\lambda)(g)$, the only if part is proved. Take any L-fuzzy point x_λ in X and $g \in L^Y$. Then

$$\begin{aligned} \mathcal{N}_{t'}(f(x)_\lambda)(g) &= \bigvee_{\substack{h\in t'\\ h\le g}} \gamma_\lambda(h(f(x)) \\ &\le \bigvee_{\substack{k\in t\\ k\le f}} \gamma_\lambda(h(x)) = \mathcal{N}_t(x_\lambda)(g\circ f), \end{aligned}$$

since $h \circ f \in t$ and $h \circ f \le g \circ f$ for all $h \in t'$ such that $h \le g$. So f is continuous.

4. γ-limit space

Let X be a set and $\mathcal{X}$ be a set of all L-fuzzy points in X.

Definition 4.1 A *γ-limit structure* is a map

$$q : \mathcal{X} \longrightarrow \mathcal{P}(\mathcal{F}_\gamma(X))$$

subject to the following conditions : $\mathcal{M} \underset{q}{\to} x_\lambda$ means $\mathcal{M} \in q(x_\lambda)$,
(γL0) $\mathcal{M} \underset{q}{\to} x_\lambda$ implies $\mathcal{M}(\overline{\alpha}) \ge \gamma_\lambda(\alpha)$ for all $\alpha \in L$,
(γL1) $\dot{x}_\lambda \underset{q}{\to} x_\lambda$ for all $x_\lambda \in \mathcal{X}$,
(γL2) if $\mathcal{M} \underset{q}{\to} x_\lambda$ and $\mathcal{M} \le \mathcal{N}$, then $\mathcal{N} \underset{q}{\to} x_\lambda$,
(γL3) if $\mathcal{M}$, $\mathcal{N} \underset{q}{\to} x_\lambda$, then $\mathcal{M} \wedge \mathcal{N} \underset{q}{\to} x_\lambda$, where $\mathcal{M} \wedge \mathcal{N}$ is defined pointwise.
The pair (X,q) is called a *γ-limit space.*

Example 4.2 (i) In the case $L = L' = \{0,1\}$, the γ-limit structure is a limitierung of filters to usual points.
(ii) In the case $L = [0,1]$ and $L' = \{0,1\}$, the γ-limit structure is a fuzzy limitierung [6] of prefilters to fuzzy points.
(iii) In the case $L = L'$ is an arbitrary completely distributive chain and $\gamma : L \to L' = id_L$, the convergence structure [4] of L-fuzzy filters to usual points induced by the L-fuzzy neighborhood filter in a stratified L-fts is in fact a γ-limit structure.

Definition 4.3 Given γ-limit spaces (X,q) and (Y,q'), the map $f : X \to Y$ is called *γ-continuous* if $\mathcal{M} \underset{q}{\to} x_\lambda$ implies $f(\mathcal{M}) \underset{q'}{\to} f(x)_\lambda$ for all $x_\lambda \in \mathcal{X}$.

Clearly, the identity map and the composition of two γ-continuous maps are γ-continuous. We note that every constant map is γ-continuous since for any constant map $X \xrightarrow{[y]} Y$ with value $y \in Y$ and $\mathcal{M} \in \mathcal{F}_\gamma(X)$ such that $\mathcal{M} \to x_\lambda$ in X, we have $[y](\mathcal{M})(g) = \mathcal{M}(g \circ [y]) = \mathcal{M}(\overline{g(y)}) \geq \gamma_\lambda(g(y)) = y_\lambda(g)$ for all $g \in L^Y$.

Let γ**Lim** be the category of all γ-limit spaces and all γ-continuous maps.

Theorem 4.4 *The category γ**Lim** is a topological category.*

Proof. For any family $((X_i, q_i))_{i \in I}$ in γ**Lim** and any source

$$(X \xrightarrow{f_i} (X_i, q_i))_{i \in I},$$

define a map $q : \mathcal{X} \longrightarrow \mathcal{P}(\mathcal{F}_\gamma(X))$ by

$$\mathcal{M} \in q(x_\lambda) \text{ iff } f_i(\mathcal{M}) \in q_i(f_i(x)_\lambda) \text{ for all } i \in I.$$

Then q is a γ-limit structure and for each $i \in I$ the map $f_i : (X, q) \to (X_i, q_i)$ is γ-continuous and for any γ-limit space (Y, q') and for any map $g : Y \to X$ such that $f_i \circ g : (Y, q') \to (X_i, q_i)$ is γ-continuous for each $i \in I$, $g : (Y, q') \to (X, q)$ is a γ-continuous map and hence it proves the initiality of q. The remainder can be easily checked.

Based on the notion of the initial γ-limit structure, we can define subspaces and products and the existence of initial structures guarantees that of final structures. Here we present an explicit form of the final γ-limit structure without proof.

Proposition 4.5 *For any family $((X_i, q_i))_{i \in I}$ in γ**Lim** and any sink*

$$((X_i, q_i) \xrightarrow{f_i} X)_{i \in I},$$

define $q : \mathcal{X} \longrightarrow \mathcal{P}(\mathcal{F}_\gamma(X))$ by $\mathcal{M} \in q(x_\lambda)$ iff $\mathcal{M} \geq \dot{x}_\lambda$ or $\mathcal{M} \geq \bigwedge_{k=1}^n f_{i_k}(\mathcal{M}_k)$, where $\mathcal{M}_k \underset{q_{i_k}}{\longrightarrow} (x_k)_\lambda$ for some $x_k \in f_{i_k}{}^{-1}(x)$, $i_k \in I$ and $k = 1, \cdots, n$. Then q is the final γ-limit structure on X.

On the basis of the notion of the final γ-limit structure, we can define quotient spaces and coproducts.

For any γ-limit spaces (X, q) and (Y, q'), let $C(X, Y)$ be the set of all γ-continuous maps from X to Y.

For any $\theta \in L^{C(X,Y)}$ and $k \in L^X$, define a map $\theta(k) : Y \to L$ by

$$y \longmapsto \theta(k)(y) = \bigvee_{\substack{f(x)=y \\ x \in X,\, f \in C(X,Y)}} (\theta(f) \wedge k(x))$$

For any γ-fuzzy filters $\mathcal{K}$ and $\mathcal{M}$ on $C(X, Y)$ and X, respectively define $\mathcal{K}(\mathcal{M}) : L^Y \to L'$ by

$$g \longmapsto \mathcal{K}(\mathcal{M})(g) = \bigvee_{\substack{\theta(k) \leq g \\ k \in L^X,\, \theta \in L^{C(X,Y)}}} (\mathcal{K}(\theta) \wedge \mathcal{M}(k)).$$

We note that if $\theta(k) \leq \overline{\alpha}$ then for any $f \in C(X, Y)$ and $x \in X$, $\theta(f) \wedge k(x) \leq \alpha$ and it implies $\theta \leq \overline{\alpha}$ or $k \leq \overline{\alpha}$. So $\mathcal{K}(\mathcal{M})(\overline{\alpha}) \leq \gamma(\alpha)$ and since L' is completely distributive $\mathcal{K}(\mathcal{M})(g \wedge h) = \mathcal{K}(\mathcal{M})(g) \wedge \mathcal{K}(\mathcal{M})(h)$ for all $g, h \in L^Y$. So $\mathcal{K}(\mathcal{M})$ is a γ-fuzzy filter on Y.

Let $\mathcal{C}(X, Y)$ be the set of all L-fuzzy points in $C(X, Y)$ and define a map $c : \mathcal{C}(X, Y) \to \mathcal{P}(\mathcal{F}_\gamma(C(X, Y)))$ as follows: for any $\mathcal{K} \in \mathcal{F}_\gamma(C(X, Y))$ and $f_\lambda \in \mathcal{C}(X, Y)$,

$$\mathcal{K} \underset{c}{\to} f_\lambda \text{ iff } \begin{cases} (1)\ \mathcal{K}(\overline{\alpha}) \geq \gamma_\lambda(\alpha) \\ (2) \text{ for any } x_\lambda \in \mathcal{X} \text{ and } \mathcal{M} \underset{q}{\to} x_\lambda, \\ \qquad \mathcal{K}(\mathcal{M}) \underset{q'}{\to} f(x)_\lambda. \end{cases}$$

Proposition 4.6 *c is a γ-limit structure on $C(X, Y)$ with respect to the evaluation map $ev : X \times C(X, Y) \to Y$ is γ-continuous.*

Proof. By the definition of c, (γL0) is trivial and for (γL1) take any $f_\lambda \in \mathcal{C}(X, Y)$. Then $\dot{f}_\lambda(\overline{\alpha}) = \gamma_\lambda(\overline{\alpha}(f)) = \gamma_\lambda(\alpha)$ and for any L-fuzzy point x_λ in X and $\mathcal{M} \underset{q}{\to} x_\lambda$,

$$\begin{aligned} &\dot{f}_\lambda(\mathcal{M})(g) \\ &= \bigvee_{\substack{\theta(k) \leq g \\ k \in L^X,\, \theta \in L^{C(X,Y)}}} \left(\dot{f}_\lambda(\theta) \wedge \mathcal{M}(k)\right) \\ &= \bigvee_{\substack{\theta(k) \leq g \\ k \in L^X,\, \theta \in L^{C(X,Y)}}} (\gamma_\lambda(\theta(f)) \wedge \mathcal{M}(k)) \end{aligned}$$

for all $g \in L^Y$. Take $\theta = f_{1_L}$ and $k = g \circ f$, then

$$\begin{aligned} \theta(k)(y) &= f_{1_L}(g \circ f)(y) \\ &= \bigvee_{\substack{m(x)=y \\ x \in X,\, m \in C(X,Y)}} (f_{1_L}(m) \wedge (g \circ f)(x)) \\ &= \bigvee_{x \in f^{-1}(y)} (g \circ f)(x) = g(y) \end{aligned}$$

and

$$\gamma_\lambda(\theta(f)) \wedge \mathcal{M}(k) = \gamma_\lambda(1_L) \wedge \mathcal{M}(g \circ f) = \mathcal{M}(g \circ f).$$

So $\dot{f}_\lambda(\mathcal{M})(g) \geq \mathcal{M}(g \circ f) = f(\mathcal{M})(g)$ for all $g \in L^Y$ and since $f(\mathcal{M}) \underset{q'}{\longrightarrow} f(x)_\lambda$, we have $\dot{f}_\lambda \underset{c}{\to} f_\lambda$. ($\gamma$L2) and ($\gamma$L3) are routine. In all, c is a γ-limit structure on $C(X, Y)$. To show that ev is

γ-continuous, let $\Phi \to (x,f)_\lambda$ in $X \times C(X,Y)$. Then $\pi_1(\Phi) \underset{q}{\to} x_\lambda$, $\pi_2(\Phi) \underset{c}{\to} f_\lambda$, where π_1 and π_2 are the first and second projection maps from $X \times C(X,Y)$ to X and $C(X,Y)$ respectively. Thus $\pi_2(\Phi)(\pi_1(\Phi)) \underset{q'}{\longrightarrow} f(x)_\lambda$. So it is enough to show that $ev(\Phi) \geq \pi_2(\Phi)(\pi_1(\Phi))$. For any $g \in L^Y$,

$$\begin{aligned}
&\pi_2(\Phi)(\pi_1(\Phi))(g)\\
&= \bigvee_{\substack{\theta(k)\leq g\\ k\in L^X,\, \theta\in L^{C(X,Y)}}} (\pi_2(\Phi)(\theta) \wedge \pi_1(\Phi)(k))\\
&= \bigvee_{\substack{\theta(k)\leq g\\ k\in L^X,\, \theta\in L^{C(X,Y)}}} (\Phi(\theta \circ \pi_2) \wedge \Phi(k \circ \pi_1))\\
&= \bigvee_{\substack{\theta(k)\leq g\\ k\in L^X,\, \theta\in L^{C(X,Y)}}} \Phi(\theta \circ \pi_2 \wedge k \circ \pi_1)
\end{aligned}$$

and for any $k \in L^X$ and $\theta \in L^{C(X,Y)}$ such that $\theta(k) \leq g$, $(\theta \circ \pi_2 \wedge k \circ \pi_1)(x,h) = \theta(h) \wedge k(x) \leq \theta(k)(h(x)) \leq g(h(x)) = (g \circ ev)(x,h)$ for all $(x,h) \in X \times C(X,Y)$. So $\pi_2(\Phi)(\pi_1(\Phi)) \leq ev(\Phi)$ and it is proved.

From now on, we assume that $C(X,Y)$ is equipped with the γ-limit structure c in Proposition 4.6, which is called the *continuous γ-limit structure* on $C(X,Y)$.

Proposition 4.7 *Let (X,q), (Y,q') and (Z,q'') be γ-limit spaces and $f : X \times Z \to Y$ be a γ-continuous map. Then there exists a unique γ-continuous map $\bar{f} : Z \to C(X,Y)$ such that $ev \circ (1_X \times \bar{f}) = f$.*

Proof. Define a map $\bar{f} : Z \to C(X,Y)$ by $\bar{f}(z)(x) = f(x,z)\ (x \in X, z \in Z)$. Then for each $z \in Z$,

$$\bar{f}(z) : X \xrightarrow{(1_X,[z])} X \times Z \xrightarrow{f} Y$$

is γ-continuous, where $[z] : X \to Z$ is a constant map with value z. To show that $\bar{f}$ is γ-continuous, let $\mathcal{L} \underset{q''}{\longrightarrow} z_\lambda$. Then $\bar{f}(\mathcal{L})(\overline{\alpha}) = \mathcal{L}(\overline{\alpha} \circ \bar{f}) = \mathcal{L}(\overline{\alpha}) \geq \gamma_\lambda(\alpha)$. Let $\mathcal{M} \underset{q}{\to} x_\lambda$. Then for any $g \in L^Y$,

$$\begin{aligned}
\bar{f}(\mathcal{M})(g) &= \bigvee_{\substack{\theta(k)\leq g\\ k\in L^X,\, \theta\in L^{C(X,Y)}}} (\bar{f}(\mathcal{L})(\theta) \wedge \mathcal{M}(k))\\
&= \bigvee_{\substack{\theta(k)\leq g\\ k\in L^X,\, \theta\in L^{C(X,Y)}}} (\mathcal{L}(\theta \circ \bar{f}) \wedge \mathcal{M}(k)).
\end{aligned}$$

Define $\mathcal{M} \times \mathcal{L} : L^{X \times Z} \to L'$ by

$$\begin{aligned}
h \longmapsto\ & (\mathcal{M} \times \mathcal{L})(h)\\
&= \bigvee_{\substack{u\circ\pi_1 \wedge v\circ\pi_2 \leq h\\ u\in L^X,\, v\in L^Z}} (\mathcal{M}(u) \wedge \mathcal{L}(v))
\end{aligned}$$

where $\pi_1(h)(x) = \vee_{z\in Z} h(x,z)$, $\pi_2(h)(z) = \vee_{x\in X} h(x,z)$. Then $\mathcal{M} \times \mathcal{L}$ is a γ-fuzzy filter on $X \times Z$. Take $u = k$ and $v = \overline{1_L}$, then $u \circ \pi_1 \wedge v \circ \pi_2 \leq k \circ \pi_1$ and $\mathcal{M}(u) \wedge \mathcal{L}(v) = \mathcal{M}(k) \wedge \mathcal{L}(\overline{1_L}) \geq \mathcal{M}(k)$ since $\mathcal{L} \underset{q''}{\longrightarrow} z_\lambda$. So $\pi_1(\mathcal{M} \times \mathcal{L}) \geq \mathcal{M}$ and hence $\pi_1(\mathcal{M} \times \mathcal{L}) \underset{q}{\to} x_\lambda$. Similarly, $\pi_2(\mathcal{M} \times \mathcal{L}) \underset{q''}{\longrightarrow} z_\lambda$. Therefore $\mathcal{M} \times \mathcal{L} \underset{q\times q''}{\longrightarrow} (x,z)_\lambda$ and since f is continuous, $f(\mathcal{M} \times \mathcal{L}) \underset{q'}{\to} f(x,z)$ and hence it suffices to show that $\bar{f}(\mathcal{L})(\mathcal{M}) \geq f(\mathcal{M} \times \mathcal{L})$. We note that for any $g \in L^Y$,

$$\begin{aligned}
f(\mathcal{M} \times \mathcal{L})(g) &= (\mathcal{M} \times \mathcal{L})(g \circ f)\\
&= \bigvee_{\substack{u\circ\pi_1 \wedge v\circ\pi_2 \leq g\circ f\\ u\in L^X,\, v\in L^Z}} (\mathcal{M}(u) \wedge \mathcal{L}(v))
\end{aligned}$$

and

$$\begin{aligned}
&\bar{f}(\mathcal{L})(\mathcal{M})(g)\\
&= \bigvee_{\substack{\theta(k)\leq g\\ k\in L^X,\, \theta\in L^{C(X,Y)}}} (\bar{f}(\mathcal{L})(\theta) \wedge \mathcal{M}(k))\\
&= \bigvee_{\substack{\theta(k)\leq g\\ k\in L^X,\, \theta\in L^{C(X,Y)}}} (\mathcal{L}(\theta \circ \bar{f}) \wedge \mathcal{M}(k)).
\end{aligned}$$

Now, for any $u \in L^X$, $v \in L^Z$ such that $u \circ \pi_1 \wedge v \circ \pi_2 \leq g \circ f$, let $k = u$ and $\theta = \bar{f}(v)$. Then

$$\begin{aligned}
&\theta(k)(y)\\
&= \bigvee_{\substack{m(x)=y\\ x\in X,\, m\in C(X,Y)}} (\theta(m) \wedge u(x))\\
&= \bigvee_{\substack{m(x)=y\\ x\in X,\, m\in C(X,Y)}} \left(\left(\bigvee_{z\in \bar{f}^{-1}(m)} v(z)\right) \wedge u(x)\right)\\
&= \bigvee_{\substack{m(x)=y\\ x\in X,\, m\in C(X,Y)}} \bigvee_{z\in \bar{f}^{-1}(m)} (v(z) \wedge u(x))\\
&\leq \bigvee_{\substack{f(x,z)=y\\ x\in X,\, z\in Z}} g(f(x,z)) = g(y)
\end{aligned}$$

for all $y \in Y$. Furthermore, since $\theta \circ \bar{f} = \bar{f}(v) \circ \bar{f} \geq v$, $\mathcal{L}(\theta \circ \bar{f}) \wedge \mathcal{M}(k) \geq \mathcal{L}(v) \wedge \mathcal{M}(u)$ and hence it is proved. Obviously $ev \circ (1_X \times \bar{f}) = f$ and such a map $\bar{f}$ is unique.

Theorem 4.8 *The category γ**Lim** is cartesian closed.*

5. The relation between LFTop and γLim

For any L-fuzzy topology t on X,

$$t^* = \{f \in L^X \ : \ f = \bigvee_{(g,\alpha)\in F} (g \wedge \overline{\alpha}) \text{ for some } F \subseteq t \times L\}$$

is the coarsest stratified L-fuzzy topology on X finer than t and for any L-fts's (X, s) and (Y, t) if $f : (X, s) \to (Y, t)$ is continuous, then $f : (X, s^*) \to (Y, t^*)$ is also continuous.

Theorem 5.1 *L**FTop*** is a bicoreflective subcategory of L**FTop**.*

Proof. For any L-fts (X, t), $(X, t^*) \xrightarrow{1_X} (X, t)$ is a bicoreflection.

Proposition 5.2 *For any stratified L-fts (X, t), define a map $q_t : \mathcal{X} \longrightarrow \mathcal{P}(\mathcal{F}_\gamma(X))$ by*

$$x_\lambda \longmapsto \quad q_t(x_\lambda) = \{\mathcal{M} \in \mathcal{F}_\lambda(X) : \mathcal{M} \geq \mathcal{N}_t(x_\lambda)\}.$$

Then (X, q_t) is a γ-limit space.

Proof. If $\mathcal{M} \underset{q_t}{\longrightarrow} x_\lambda$, then $\mathcal{M}(\overline{\alpha}) \geq \mathcal{N}_t(x_\lambda)(\overline{\alpha}) = \gamma_\lambda(\text{int}\,\overline{\alpha}(x)) = \gamma_\lambda(\alpha)$. So ($\gamma$L0) is fulfilled and the remainder is routine.

Proposition 5.3 *For any stratified L-fts's (X, t), (Y, t') if $f : (X, t) \to (Y, t')$ is continuous, then $f : (X, q_t) \to (Y, q_{t'})$ is γ-continuous.*

Proof. It follows immediately from the fact that if $\mathcal{M} \geq \mathcal{N}_t(x_\lambda)$, then $f(\mathcal{M})(g) = \mathcal{M}(g \circ f) \geq \mathcal{N}_t(x_\lambda)(g \circ f) \geq \mathcal{N}_{t'}(f(x)_\lambda)(g)$.

Proposition 5.4 *For any γ-limit space (X, q), define*

$$t_q = \{f \in L^X \ : \ \text{for any } x_\lambda \in \mathcal{X} \text{ and } \beta \in L' \ \ \gamma_\lambda(f(x)) > \beta \text{ and } \mathcal{M} \underset{q}{\to} x_\lambda \ \ \text{imply } \mathcal{M}(f) > \beta\}.$$

Then t_q is a stratified L-fuzzy topology on X.

Proof. For each $\alpha \in L$, $\overline{\alpha} \in t_q$, since $\gamma_\lambda(\overline{\alpha}(x)) = \gamma_\lambda(\alpha) > \beta$ and $\mathcal{M} \underset{q}{\to} x_\lambda$ imply $\mathcal{M}(\overline{\alpha}) \geq \gamma_\lambda(\alpha) > \beta$. The remainder is immediate from $\vee_{i\in I} f_i(x) > \lambda^*$ iff $f_i(x) > \lambda^*$ for some $i \in I$ and $(f_1 \wedge f_2)(x) > \lambda^*$ iff $f_1(x) > \lambda^*$ and $f_2(x) > \lambda^*$.

Proposition 5.5 *For any γ-limit spaces (X, q), (Y, q') if $f : (X, q) \to (Y, q')$ is γ-continuous, then $f : (X, t_q) \to (Y, t_{q'})$ is continuous.*

Proof. Let $g \in t_{q'}$. Suppose $\gamma_\lambda((g \circ f)(x)) > \beta$ for some L-fuzzy point x_λ in X and $\beta \in L'$ and $\mathcal{M} \underset{q}{\to} x_\lambda$. Then $\gamma_\lambda(g(f(x)) > \beta$ and $f(\mathcal{M}) \underset{q'}{\longrightarrow} f(x)_\lambda$, consequently we have $f(\mathcal{M})(g) = \mathcal{M}(g \circ f) > \beta$.

Proposition 5.6 *For any γ-limit space (X, q), $q(x_\lambda) \subseteq q_{t_q}(x_\lambda)$ for all $x_\lambda \in \mathcal{X}$.*

Proof. Take any $\mathcal{M} \in \mathcal{F}_\gamma(X)$ and $x_\lambda \in \mathcal{X}$ and suppose $\mathcal{M} \underset{q}{\to} x_\lambda$. In case $\mathcal{N}_{t_q}(x_\lambda)(f) = 0'_L$, it is clear. If $\mathcal{N}_{t_q}(x_\lambda)(f) > 0_{L'}$, then for any $\beta \in L'$ such that $\mathcal{N}_{t_q}(x_\lambda)(f) > \beta$ there exists $g \in t_q$ such that $g \leq f$ and $\gamma_\lambda(g(x)) > \beta$. So we have $\mathcal{M}(g) > \beta$ and since $\mathcal{M}$ preserves the order of L^X, $\mathcal{M}(f) \geq \mathcal{N}_{t_q}(x_\lambda)(f)$ and thus $\mathcal{M} \underset{q_{t_q}}{\longrightarrow} x_\lambda$.

Proposition 5.7 *For any stratified L-fts (X, t), $t = t_{q_t}$.*

Proof. Take any $f \in t$. Suppose $\gamma_\lambda(f(x)) > \beta$ and $\mathcal{M} \underset{q_t}{\longrightarrow} x_\lambda$ for some $\mathcal{M} \in \mathcal{F}_\gamma(X)$, $x_\lambda \in \mathcal{X}$ and $\beta \in L'$. Then since

$$\mathcal{M}(f) \geq \mathcal{N}_t(x_\lambda)(f) = \gamma_\lambda(\text{int}f(x)) = \gamma_\lambda(f(x)) > \beta,$$

we have $f \in t_{q_t}$. Conversely, take any $f \in t_{q_t}$, then for any L-fuzzy point x_λ in X and $\beta \in L'$ such that $\gamma_\lambda(f(x)) > \beta$ and $\mathcal{M} \underset{q_t}{\longrightarrow} x_\lambda$, it holds $\mathcal{M}(f) > \beta$. So for any L-fuzzy point x_λ in X and $\beta \in L'$ such that $\gamma_\lambda(f(x)) > \beta$, $\mathcal{N}_t(x_\lambda)(f) > \beta$ and hence we have $f \in t$.

Theorem 5.8 *L**FTop*** is a bireflective subcategory of γ**Lim**.*

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