FUZZY L-CONVERGENCE SPACES

Kyung Chan MIN

Dept. of Mathematics, Yonsei University

Seoul, 120-749 Korea

Tel:82-2-361-2591 Fax:82-2-392-6634 E-mail:kcmin@bubble.yonsei.ac.kr

Abstract

A notion of 'fuzzy' convergence of filters on a set is introduced. We show that the collection of fuzzy L-limit spaces forms a cartesian closed topological category and obtain an interesting relationship between the notions of 'fuzzy' convergence structure and convergence approach spaces.

Keywords : fuzzy L-limitierung, cartesian closed, convergence M-approach structure.

1. Introduction

The gap between mathematical model and physical reality was one of main concern to many researchers for a long time. K.Menger was the first to consider this problem, consequently introducing the concept of statistical (probabilistic) metric spaces [11]. Since then, probabilistic metric and topological spaces have been studied extensively. In probability, statistics and analysis, many types of convergence was known to be non-topological. In 1989, Florescu [6] initiated a sutdy of 'probabilistic convergence structures' based on net convergence. Richardson and Kent [14] introduced a filter formulation of this theory. Herrlich and Zhang [8] studied some categorical properties of probabilistic convergence spaces. On the other hand, R.Lowen [10] introduced a notion of approach space as a generalization of the notions of topological space and metric space and E.Lowen and R.Lowen [9] introduced a notion of convergence approach space understanding convergence spaces and metric spaces as entities of the same kind.

Many researchers studied on the convergence of prefilters or fuzzy filters on a set. (see [4,12]) In this paper we introduce a notion of 'fuzzy' convergence of filters on a set as a generalization of probabilistic convergence structure in [8] and obtain some basic categorical properties. Moreover, we show that the notions of 'fuzzy' convergence structure and convergence approach structure are equivalent in a certain case.

2. Fuzzy L-limitierung

Let L be a complete Heyting algebra with a top element 1 and a bottom element 0. For a set X, $F(X)$ is the collection of all filters on X.

Definition 2.1. Let X be a set. A map $l : F(X) \times X \to L$ is called a *fuzzy L-limitierung* on X if it satisfies the following conditions:

(1) $l(\dot{x}, x) = 1$ for all $x \in X$.

(2) if $\mathcal{F} \subseteq \mathcal{G}$, then $l(\mathcal{F}, x) \leq l(\mathcal{G}, x)$ for all $x \in X$.

(3) for $\mathcal{F}, \mathcal{G} \in F(X), l(\mathcal{F}, x) \wedge l(\mathcal{G}, x) \leq l(\mathcal{F} \cap \mathcal{G}, x)$.

The pair (X, l) is called a *fuzzy L-limit space.*

Definition 2.2. A map $f : (X, l) \to (Y, m)$ between fuzzy L-limit spaces is said to be *continuous* if

$$l(\mathcal{F}, x) \leq m(f(\mathcal{F}), f(x))$$

for all $(\mathcal{F}, x) \in F(X) \times X$.

We form a category $\underline{FLlim}$ consisting of all fuzzy L-limit spaces and all continuous maps between them.

Remark. 1. If $L = \{0, 1\}$, then $\underline{FLlim} = \underline{Lim}$, the category of limit spaces and continuous maps.

2. If $L = [0, 1]$, a fuzzy L-limit space means a probabilistic limit spaces [4].

Theorem 2.3. *The category $\underline{FLim}$ is topological.*

Proof. Let X be a set and $(X_\alpha, l_\alpha) \in \underline{FLlim}$ for each $\alpha \in \Lambda$. Let $f_\alpha : A \to (X_\alpha, l_\alpha)$ be a map for each $\alpha \in \Lambda$. Define $l : F(X) \times X \to L$ by $l(\mathcal{F}, x) = \inf_\lambda l_\alpha(f_\alpha(\mathcal{F}), f(x))$. Then $l(\dot{x}, x) = 1$. Let $\mathcal{F} \subseteq \mathcal{G}$ in $f(X)$. Then

$$\begin{aligned} l(\mathcal{F}, x) &= \inf_\lambda l_\alpha(\mathcal{F}, f_\alpha(x)) \\ &\leq \inf_\lambda l_\alpha(f_\alpha(\mathcal{G}), f_\alpha(x)) \\ &= l(\mathcal{G}, x). \end{aligned}$$

Hence $(X, l) \in \underline{FLlim}$.

Let $(Z, m) \in \underline{FLlim}$ and $g : Z \to X$ be a map. Suppose $f_\alpha \circ g$ is continuous for all $x \in \Lambda$. Then $m(\mathcal{A}, z) \leq l_\alpha(f_\alpha(g(\mathcal{A})), f_\alpha(g(z)))$ for each $(\mathcal{A}, z) \in F(Z) \times Z$ and $\alpha \in \Lambda$. Hence $m(\mathcal{A}, z) \leq l(g(\mathcal{A}), g(z))$.

We note that the category $\underline{FLlim}$ satisfies the fibre small condition and it has the terminal separator property.

Remark 1. Let $\{(X_i, l_i)\}_I$ be a family of fuzzy L-limit spaces. Let $l : F(\prod_I X_i) \times \prod_I X_i \to L$ be a map defined by

$$l(\mathcal{H}, (x_i)) = l_i(\prod_i \pi_i(\mathcal{H}), x_i).$$

Then l is the fuzzy L-limitierung for the product space $\prod_I X_i$ in $\underline{FLlim}$.

Theorem 2.4. *The category* $\underline{FLlim}$ *is cartesian closed.*

Proof. For (X, l) and (Y, m) in $\underline{FLlim}$, let $C(X, Y)$ be the set of all continuous maps between them. For each $\mathcal{L} \in F(C(X, Y))$ an $f \in C(X, Y)$, let $u(\mathcal{L}, f) = \sup\{\alpha \in L | l(\mathcal{A}, x) \wedge \alpha \leq m(\mathcal{L}(\mathcal{A}), f(x))$ for all $(\mathcal{A}, x) \in F(X) \times X\}$. Note that $l(\mathcal{A}, x) \wedge u(\mathcal{L}, f) \leq m(\mathcal{L}(\mathcal{A}, x))$ for all $(\mathcal{A}, x) \in F(X) \times X$. Clearly, $u(\dot{f}, f) = 1$. Let $\mathcal{L}_1 \subseteq \mathcal{L}_2 \in F(C(X, Y))$, then from the formula $l(\mathcal{A}, x) \wedge \alpha \leq m(\mathcal{L}_1(\mathcal{A}), f(x)) \leq m(\mathcal{L}_2(\mathcal{A}), f(x))$, it is clear to see that $u(\mathcal{L}_1, f) \leq u(\mathcal{L}_2, f)$ for all $f \in C(X, Y)$.

Let $\mathcal{L}_1, \mathcal{L}_2 \in F(C(X, Y))$ and $f \in C(X, Y)$. Let $\alpha_1, \alpha_2 \in L$ satisfying the following

$$l(\mathcal{A}, x) \wedge \alpha_1 \leq m(\mathcal{L}_1(\mathcal{A}), f(x))$$

and

$$l(\mathcal{A}, x) \wedge \alpha_2 \leq m(\mathcal{L}_2(\mathcal{A}), f(x))$$

for all $(\mathcal{A}, x) \in F(X) \times X$. Then

$$\begin{aligned} l(\mathcal{A}, x) \wedge (\alpha_1 \wedge \alpha_2) &\leq m(\mathcal{L}_1(\mathcal{A}), f(x)) \wedge m(\mathcal{L}_2(\mathcal{A}), f(x)) \\ &\leq m(\mathcal{L}_1(\mathcal{A}) \cap \mathcal{L}_2(\mathcal{A}), f(x)) \\ &= m((\mathcal{L}_1 \cap \mathcal{L}_2)(\mathcal{A}), f(x)) \end{aligned}$$

for all $(\mathcal{A}, x) \in F(X) \times X$. Therefore it is easy to check that $u(\mathcal{L}_1, f) \wedge u(\mathcal{L}_2, f) \leq u(\mathcal{L}_1 \cap \mathcal{L}_2, f)$. Therefore $(C(X, Y), u) \in \underline{FLlim}$.

Let $\mathcal{H} \in F(X \times C(X,Y))$ and k be the fuzzy L-limitierung for the product space $X \times C(X,Y)$. Then

$$\begin{aligned} k(\mathcal{H},(x,f)) &= l(\pi_1(\mathcal{H}),x) \wedge u(\pi_2(\mathcal{H}),f) \\ &\leq m(\pi_2(\mathcal{H})(\pi_1(\mathcal{A})),f(x)) \\ &\leq m(ev(\mathcal{H}),f(x)). \end{aligned}$$

Hence $ev : X \times C(X,Y) \to Y$ is continuous. Let $h : (X,l) \times (Z,m) \to (Y,m)$ be a map in $\underline{FLlim}$. Define $h^* : (Z,m) \to (C(X,Y),u)$ by $h^*(z)(x) = h(x,z)$. Since $\underline{FLlim}$ has the terminal separator property, $h^*(z) \in C(X,Y)$ for all $z \in Z$. Take any $(\mathcal{G},z) \in F(Z) \times Z$ and $(\mathcal{A},x) \in F(X) \times X$. Since h is continuous,

$$\begin{aligned} l(\mathcal{A},x) \wedge n(\mathcal{G},z) &\leq m(h(\mathcal{A} \times \mathcal{G}),h(x,z)) \\ &= m(h^*(\mathcal{G})(\mathcal{A}),h^*(z)(x)). \end{aligned}$$

Hence $n(\mathcal{G},z) \leq u(h^*(\mathcal{G}),h^*(z))$. Therefore h^* is continuous. The uniqueness of such a map h^* is obvious.

If we take $L = [0,1]$, then we have the following.

Corollary 2.5. *[8] The category $\underline{P\text{−}Lim}$ of probabilistic limit spaces is a cartesian closed topological category.*

3. Convergence M-approach spaces

Let M be a complete Heyting algebra.

Definition 3.1. A *convergence M-approach limit* λ on a set X is a function $\lambda : \mathcal{F}(X) \to M^X$ satisfying the following conditions:

A1) For any $x \in X$, $\lambda(\dot{x})(x) = 0$.

A2) If $\mathcal{F} \subseteq \mathcal{G}$, then $\lambda(\mathcal{G}) \leq \lambda(\mathcal{F})$.

A3) For all $\mathcal{F},\mathcal{G} \in \mathcal{F}(X)$, $\lambda(\mathcal{F} \cap \mathcal{G}) \leq \lambda(\mathcal{F}) \vee \lambda(\mathcal{G})$.

The pair (X,λ) is called a *convergence M-approach space.*

Definition 3.2. A map $f : (X,\lambda) \to (Y,\delta)$ between convergence M-approach space is called a *contraction* if $\delta(f(\mathcal{F})) \circ f \leq \lambda(\mathcal{F})$.

We form a category CMAP of all convergence M-approach spaces and all contractions.

Note that a convergence M-approach limit λ on a set X can be considered as a map $\lambda : \mathcal{F}(X) \times X \to M$ if we put $\lambda(\mathcal{F}, x) = \lambda(\mathcal{F})(x)$ for each $(\mathcal{F}, x) \in \mathcal{F}(X) \times X$. Hence it is easy to see that the notion of a fuzzy L-limitierung on a set X is equivalent to the notion of a convergence M-approach limit if there is an order-reversing isomorphism between L and M. Therefore we have the following.

Theorem 3.3. *The category $\underline{CMAP}$ is isomorphic to the category $\underline{FLlim}$, if there is an order-reversing isomorphism between M and L.*

Clearly, there are many interesting order-reversing isomorphism between $[0, \infty]$ and $[0, 1]$.

Corollary 3.4. *[1] The category $\underline{CAP}$ of convergence approach spaces and the category $\underline{P-Lim}$ of probabilitic limit spaces are isomorphic.*

Remarks 1. The numerical scales $[0, 1]$ and $[0, \infty]$ have some interesting relationships. In fact, they generate the notions of fuzzy set and toll set [3]. Min and Park [13] showed that the category of fuzzy sets and the category of toll sets are isomorphic as a topos.

2. We note that the category CAP is a quasitopos containings the categories $CONV$, of convergence spaces, and MET, of metric spaces as nicely embedded subcategories. (See [9]). On a separate paper we will discuss about whether the categories $FLlim$ and $CMAP$ are also quasitopos and some categorical relationships among intersting categories.

References

1. P.Brock and D.C.Kent, "Approach spaces, Limit tower spaces, and Probabilistic Convergence spaces", Applied Categorical Structures, V.5, No.2 (1997), 99-110.
2. P.Brock and D.Kent, "On convergence approach spaces", Applied Categorical Structures, V.6, No.1 (1998), 117-125.
3. D.Dubois and H.Prade, "Tolls sets", Proc. IFSA '91, Brusels, Artificial Intelligence (1991), 21-24.
4. P.Eklund and W.Gähler, "Contributions to fuzzy convergence", Recent Developments of General Topology and its Applications, Academic Verlag (1992), 118-123.
5. H.R.Fisher, "Limesräume", Math. Ann. 137 (1959), 269-303.

6. L.C.Florescu, "Probabilistic convergence structures", Aequationes Math. 38 (1989), 123-145.
7. H.Herrlich, "Cartesian closed topological categories", Math. Collog. Univ. Cape Town 9 (1974), 1-16.
8. H.Herrlich and D.Zhang, "Categorical properties of probabilistic convergence spaces", preprint.
9. E.Lowen and R.Lowen, "A quasitiopos containing $CONV$ and MET as full subcategories", Intl. J. Math. Math. Sci. 11 (1988), 417-438.
10. R.Lowen, "Approach spaces: A common supercategory of TOP and MET", Math. Nachr. 141 (1989), 183-226.
11. K.Menger, "Stastical metrics", Proc. Nat. Acad. Sci. U.S.A. 28 (1942), 535-537.
12. K.C.Min, "Fuzzy limit spaces", Fuzzy Sets and Systems 32 (1989), 343-357.
13. K.C.Min and J.W.Park, "Relations between toll sets and fuzzy sets", Proc. AFSS '93, Sincapore (1993).
14. G.D.Richardson and D.C.Kent, "Probabilistic convergence spaces", J. Austral. Math. Soc. (Series A) 61 (1996), 400-420.