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PRODUCTS OF T-FUZZY FINITE STATE MACHINES

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Abstract.

We introduce the concepts of coverings, direct products, cascade products and wreath products of T-fuzzy finite state machines and investigate their algebraic structures.

Keywords: fuzzy finite state machine, T-fuzzy finite state machine, covering, restricted direct product, full direct product, cascade product, wreath produc

1. Introduction

Since Wee [8] in 1967 introduced the concept of fuzzy automata following Zadeh [9], fuzzy automata theory has been developed by many researchers. Recently Malik et al. [4-6] introduced the concepts of fuzzy finite state machines and fuzzy transformation semigroups based on Wee's concept [8] of fuzzy automata and related concepts and applied algebraic technique. Cho et al. [2,3] introduced the notion of a T-fuzzy finite state machine that is an extension of a fuzzy finite state machine. Even if $T = w$, our notion is different from the notion of Malik et al. [5]. In this paper, we introduce the concepts of coverings, restricted direct products, full direct products, cascade products and wreath products of T-fuzzy finite state machines that are generalizations of crisp concepts in algebraic automata theory and investigate their algebraic structures.

For the terminology in (crisp) algebraic automata theory, we refer to [1].

2. T-fuzzy finite state machines

Definition 2.1 [3] A triple $\mathcal{M} = (Q, X, \tau)$ where Q and X are finite nonempty sets and τ is a fuzzy subset of $Q \times X \times Q$, i.e., τ is a function from $Q \times X \times Q$ to $[0,1]$, is called a fuzzy finite state machine if $\sum_{q \in Q} \tau(p,a,q) \leq 1$ for all $p \in Q$ and $a \in X$. If $\sum_{q \in Q} \tau(p,a,q) = 1$ for all $p \in Q$ and $a \in X$, then $\mathcal{M}$ is said to be complete.

Note that our notion of a fuzzy finite state machine is different from the notion of a fuzzy finite state machine of [5] that also is a generalizaiton of the notion of a (crisp) state machine.

Let $\mathcal{M} = (Q, X, \tau)$ be a fuzzy finite state machine. Then Q is called the set of states and X is called the set of input symbols. Let X^+ denote the set of all words of elements of X of finite length.

Definition 2.2 [7] A binary operation T on $[0,1]$ is called a t-norm if

(1) $T(a,1) = a$,
(2) $T(a,b) \leq T(a,c)$ whenever $b \leq c$,
(3) $T(a,b) = T(b,a)$,
(4) $T(a,T(b,c)) = T(T(a,b),c)$

for all $a,b,c \in [0,1]$.

The maximum and minimum will be written as $\vee$ and $\wedge$, respectively. T is clearly $\vee$-distributive, i.e., $T(a \vee b, c) = T(a,c) \vee T(b,c)$ for all $a,b,c \in [0,1]$. Define T_0 on $[0,1]$ by $T_0(a,1) = a = T_0(1,a)$ and $T_0(a,b) = 0$ if $a \neq 1$ and $b \neq 1$ for all $a,b \in [0,1]$. Then $\wedge$ is the greatest t-norm on $[0,1]$ and T_0 is the least t-norm on $[0,1]$, i.e., for any t-norm T, $\wedge(a,b) \geq T(a,b) \geq T_0(a,b)$ for all $a,b \in [0,1]$.

T will always mean a t-norm on $[0,1]$. By an abuse of notation we will denote $T(a_1, T(a_2, T(\cdots, T(a_{n-1}, a_n)\cdots)))$ by $T(a_1, \cdots, a_n)$ where $a_1, \cdots, a_n \in [0,1]$. The legitimacy of this abuse is ensured by the associativity of T (Definition 2.2(4)).

Definition 2.3 [3] Let $\mathcal{M} = (Q, X, \tau)$ be a fuzzy finite state machine. Define $\tau^+ : Q \times X^+ \times Q \rightarrow$

$[0,1]$ by

$$\begin{aligned} & \tau^+(p, a_1 \cdots a_n, q) \\ = & \vee\{T(\tau(p, a_1, r_1), \tau(r_1, a_2, r_2), \cdots, \\ & \tau(r_{n-2}, a_{n-1}, r_{n-1}), \tau(r_{n-1}, a_n, q)) | r_i \in Q\} \end{aligned}$$

where $p, q \in Q$ and $a_1, \cdots, a_n \in X$. When T is applied to $\mathcal{M}$ as above, $\mathcal{M}$ is called a T-fuzzy finite state machine.

Proposition 2.4 [3] *Let (Q, X, τ) be a T-fuzzy finite state machine. Then*

$$\tau^+(p, xy, q) = \vee\{T(\tau^+(p, x, r), \tau^+(r, y, q)) \mid r \in Q\}$$

for all $p, q \in Q$ and $x, y \in X^+$.

3. Coverings

Definition 3.1 Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be T-fuzzy finite state machines. If $\xi : X_1 \to X_2$ is a function and $\eta : Q_2 \to Q_1$ is a surjective partial function such that $\tau_1^+(\eta(p), x, \eta(q)) \leq \tau_2^+(p, \xi(x), q)$ for all p, q in the domain of η and $x \in X_1^+$, then we say that (η, ξ) is a covering of $\mathcal{M}_1$ by $\mathcal{M}_2$ and that $\mathcal{M}_2$ covers $\mathcal{M}_1$ and denote by $\mathcal{M}_1 \leq \mathcal{M}_2$. Moreover, if the inequality turns out equality whenever the left hand side of the inequality is not zero [resp. the inequality always turns out equality], then we say that (η, ξ) is a strong covering [resp. a complete covering] of $\mathcal{M}_1$ by $\mathcal{M}_2$ and that $\mathcal{M}_2$ strongly covers [resp. completely covers] $\mathcal{M}_1$ and denote by $\mathcal{M}_1 \leq_s \mathcal{M}_\in$ [resp. $\mathcal{M}_1 \leq_c \mathcal{M}_2$].

In Definition 3.1, we abused the function ξ. We will write the natural semigroup homomorphism from X_1^+ to X_2^+ induced by ξ by ξ also for convenience sake. We give an example that is elementary and important.

Example 3.2 Let $\mathcal{M} = (Q, X, \tau)$ be a T-fuzzy finite state machine. Define an equivalence relation $\sim$ on X by $a \sim b$ if and only if $\tau(p, a, q) = \tau(p, b, q)$ for all $p, q \in Q$. Construct a T-fuzzy finite state machine $\mathcal{M}_1 = (Q, X/\sim, \tau^\sim)$ by defining $\tau^\sim(p, [a], q) = \tau(p, a, q)$. Now define $\xi : X \to X/\sim$ by $\xi(a) = [a]$ and $\eta = 1_Q$. Then (η, ξ) is a complete covering of $\mathcal{M}$ by $\mathcal{M}_1$ clearly.

Proposition 3.3 *Let $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be T-fuzzy finite state machines. If $\mathcal{M}_1 \leq \mathcal{M}_2$ [resp. $\mathcal{M}_1 \leq_s \mathcal{M}_2$, $\mathcal{M}_1 \leq_c \mathcal{M}_\in$] and $\mathcal{M}_2 \leq \mathcal{M}_3$ [resp. $\mathcal{M}_2 \leq_s \mathcal{M}_3$, $\mathcal{M}_2 \leq_c \mathcal{M}_3$], then $\mathcal{M}_1 \leq \mathcal{M}_3$ [resp. $\mathcal{M}_1 \leq_s \mathcal{M}_3$, $\mathcal{M}_1 \leq_c \mathcal{M}_3$].*

4. Direct products

In this section, we consider restricted direct products and full direct products of T-fuzzy finite state machines, where T is less than or equal to the ordinary product. We will always assume that T is less than or equal to the ordinary product.

Definition 4.1 Let $\mathcal{M}_1 = (Q_1, X, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X, \tau_2)$ be T-fuzzy finite state machines. The restricted direct product $\mathcal{M}_1 \wedge_T \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the T-fuzzy finite state machine $(Q_1 \times Q_2, X, \tau_1 \wedge_T \tau_2)$ with

$$\begin{aligned} & (\tau_1 \wedge_T \tau_2)((p_1, p_2), a, (q_1, q_2)) \\ = & T(\tau_1(p_1, a, q_1), \tau_2(p_2, a, q_2)). \end{aligned}$$

Theorem 4.2 *Let $\mathcal{M}_1 = (Q_1, X, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X, \tau_2)$ be T-fuzzy finite state machines. Then $(\tau_1 \wedge_T \tau_2)^+((p_1, p_2), x, (q_1, q_2)) = T(\tau_1^+(p_1, x, q_1), \tau_2^+(p_2, x, q_2))$ for all $p_1, q_1 \in Q_1$, $p_2, q_2 \in Q_2$ and $x \in X^+$.*

Definition 4.3 Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be T-fuzzy finite state machines. The full direct product $\mathcal{M}_1 \times_T \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the T-fuzzy finite state machine $(Q_1 \times Q_2, X_1 \times X_2, \tau_1 \times_T \tau_2)$ with $(\tau_1 \times_T \tau_2)((p_1, p_2), (a, b), (q_1, q_2)) = T(\tau_1(p_1, a, q_1), \tau_2(p_2, b, q_2))$.

Theorem 4.4 *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be T-fuzzy finite state machines. Then*

$$\begin{aligned} & (\tau_1 \times_T \tau_2)^+((p_1, p_2), (a_1 \cdots a_n, b_1 \cdots b_n), (q_1, q_2)) \\ = & T(\tau_1^+(p_1, a_1 \cdots a_n, q_1), \tau_2^+(p_2, b_1 \cdots b_n, q_2)) \end{aligned}$$

for all $a_1, \cdots, a_n \in X_1$, $b_1, \cdots, b_n \in X_2$, $p_1, q_1 \in Q_1$ and $p_2, q_2 \in Q_2$.

Proposition 4.5 *Let $\mathcal{M}_1 = (Q_1, X, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X, \tau_2)$ be T-fuzzy finite state machines. Then $\mathcal{M}_1 \wedge_T \mathcal{M}_2 \leq_c \mathcal{M}_1 \times_T \mathcal{M}_2$.*

The following proposition is a direct consequence of the associativity of $\wedge_T$.

Proposition 4.6 *Let $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ be T-fuzzy finite state machines. Then the following are hold:*
(i) $(\mathcal{M}_1 \wedge_T \mathcal{M}_2) \wedge_T \mathcal{M}_3 = \mathcal{M}_1 \wedge_T (\mathcal{M}_2 \wedge_T \mathcal{M}_3)$.
(ii) $(\mathcal{M}_1 \times_T \mathcal{M}_2) \times_T \mathcal{M}_3 = \mathcal{M}_1 \times_T (\mathcal{M}_2 \times_T \mathcal{M}_3)$.

5. Cascade products and wreath products

In this section, we consider cascade products and wreath products of T-fuzzy finite state machines, where T is less than or equal to the ordinary product. We will always assume that T is less than or equal to the ordinary product.

Definition 5.1 Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be T-fuzzy finite state machines. The cascade product $\mathcal{M}_1 \emptyset_T \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ with respect to $\omega : Q_2 \times X_2 \to X_1$ is the T-fuzzy finite state machine $(Q_1 \times Q_2, X_2, \tau_1 \omega_T \tau_2)$ with

$$\begin{aligned}&(\tau_1 \omega_T \tau_2)((p_1, p_2), b, (q_1, q_2)) \\ = \;& T(\tau_1(p_1, \emptyset(p_2, b), q_1), \tau_2(p_2, b, q_2)).\end{aligned}$$

Theorem 5.2 *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be T-fuzzy finite state machines. Then*

$$\begin{aligned}&(\tau_1 \omega_T \tau_2)^+((p_1, p_2), x, (q_1, q_2)) \\ = \;& T(\tau_1^+(p_1, \omega^+(p_2, x), q_1), \tau_2^+(p_2, x, q_2))\end{aligned}$$

where $p_1, q_1 \in Q_1, p_2, q_2 \in Q_2$ and $x \in X_2^+$.

Definition 5.3 Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be T-fuzzy finite state machines. The wreath product $\mathcal{M}_1 \circ_T \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the T-fuzzy finite state machine $(Q_1 \times Q_2, X_1^{Q_2} \times X_2, \tau_1 \circ_T \tau_2)$ with

$$\begin{aligned}&(\tau_1 \circ_T \tau_2)((p_1, p_2), (f, b), (q_1, q_2)) \\ = \;& T(\tau_1(p_1, f(p_2), q_1), \tau_2(p_2, b, q_2)).\end{aligned}$$

Theorem 5.4 *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ are T-fuzzy finite state machines. Then*

$$\mathcal{M}_1 \omega_T \mathcal{M}_2 \leq_c \mathcal{M}_1 \circ_T \mathcal{M}_2.$$

Corollary 5.5 *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1), \mathcal{M}_2 = (Q_2, X_2, \tau_2)$ and $\mathcal{M} = (Q, X, \tau)$ are T-fuzzy finite state machines. If $\mathcal{M} \leq \mathcal{M}_1 \omega_T \mathcal{M}_2$, then $\mathcal{M} \leq \mathcal{M}_1 \circ_T \mathcal{M}_2$.*

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