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A Stabilization Algorithm for Fuzzy Systems with Singleton Consequents

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Abstract

This paper presents a stabilization algorithm for a class of fuzzy systems with singleton consequents. To this aim, we introduce two canonical forms of an unforced fuzzy system and a stability theorem. A design example is shown to verify the stabilization algorithm.

Keywords : Fuzzy Control, Model-Based Control, Stability Analysis, Optimal Control

1. Introduction

There are three types of fuzzy (control) systems classified by their consequents : fuzzy sets (type I), singletons (type II), or linear functions (type III).

Recently many studies on the model-based design of fuzzy conrollers are centered around the type III[3,4]. The idea of the stability analysis is to regard fuzziness in the type III systems, i.e., nonlinearity, as uncertainty in polytopic linear systems and embed a stability problem in robust control theory. On the otherhand, there have been few on the model-based control of the type I systems[1,2]. Most of studies on the stability of the type I control systems are concerned with the stability analysis of a certain non-fuzzy system with a fuzzy controller.

In the case of the type I and II systems, a reason for few studies on the model-based control is that there has been no theoretical study on the stability of unforced fuzzy systems. Recently, one of the authors has presented a way to stability analysis of an unforced type II fuzzy system[5]. In this paper we discuss a stabilization algorithm for the type II fuzzy systems.

2. Preliminaries

In this section, we introduce two canonical forms and a stability theorem of two-dimensional continuous fuzzy systems with singleton consequents.

Suppose a system in the following form :

$$if\ \boldsymbol{x}\ is\ G^{\sigma\tau}(\boldsymbol{x}),\ then\ \dot{\boldsymbol{x}}\ is\ \boldsymbol{h}(\sigma,\tau), \qquad (1)$$
$$\sigma = 1,2,\cdots,n_1,\ \tau = 1,2,\cdots,n_2.$$

where $\boldsymbol{x}(t) = (x_1(t), x_2(t))^T$ is a two-dimensional state vector, $G^{\sigma\tau}(\boldsymbol{x}) = (G_1^\sigma(x_1), G_2^\tau(x_2))^T$ is a membership function vector with respect to $\boldsymbol{x}$, $\boldsymbol{h}(\sigma,\tau) = (h_1(\sigma,\tau),\ h_2(\sigma,\tau))^T$ is a singleton consequent vector, $n_1, n_2 \geq 2$ and T denotes 'transpose'.

We assume that G_1^σ and G_2^τ are normalized membership functions of a triangular form as follow :

$$G_i^\lambda(x_i) = \begin{cases} \dfrac{x_i - d_i(\lambda-1)}{d_i(\lambda) - d_i(\lambda-1)}, & d_i(\lambda-1) \leq x_i \leq d_i(\lambda) \\ \dfrac{d_i(\lambda+1) - x_i}{d_i(\lambda+1) - d_i(\lambda)}, & d_i(\lambda) \leq x_i \leq d_i(\lambda+1) \\ 0, & \text{otherwise} \end{cases} \qquad (2)$$

where $i = 1, 2$, and also we assume that $d_i(\lambda) < d_i(\lambda+1)$, $\lambda = \sigma$ or τ.

Define a square $R_{\sigma\tau}$ and a vector $\boldsymbol{d}(\sigma,\tau)$ in two-dimensional space as

$$R_{\sigma\tau} \equiv [d_1(\sigma),\ d_1(\sigma+1)] \times [d_2(\tau),\ d_2(\tau+1)] \qquad (3)$$
$$\boldsymbol{d}(\sigma,\tau) \equiv (d_1(\sigma),\ d_2(\tau))^T. \qquad (4)$$

We assume that there exist σ and τ such that $d_1(\sigma) < 0 < d_1(\sigma+1)$ and $d_2(\tau) < 0 < d_2(\tau+1)$, and call this zero-square denoted by $R_{\sigma\tau}^o$. Fig. 1 shows squares allocated on the state-space. $\dot{\boldsymbol{x}}$ in (1) is inferred as

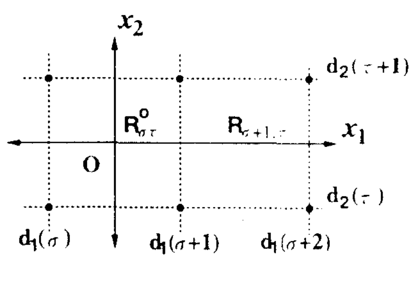

Fig. 1 Squares on the state-space (• is a vertex)

$$\dot{\boldsymbol{x}} = \sum_{i=\sigma}^{\sigma+1} \sum_{j=\tau}^{\tau+1} G_1^i(x_1(t)) G_2^j(x_2(t)) \boldsymbol{h}(i,j), \quad \boldsymbol{x} \in R_{\sigma\tau} \quad (5)$$

where $G_1^\sigma + G_1^{\sigma+1} = 1$, $G_2^\tau + G_2^{\tau+1} = 1$ and $\sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} G_1^i G_2^j = 1$. We also obtain

$$x = \sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} G_1^i(x_1(t))G_2^j(x_2(t))\boldsymbol{d}(i,j), \quad \boldsymbol{x} \in R_{\sigma\tau}. \quad (6)$$

Based on this fact, we obtain the next parametric expression.

Parametric expression[5] *For $x \in R_{\sigma\tau}$ the fuzzy system (5) is expressed as*

$$\begin{aligned} x &= \alpha_1\alpha_2\boldsymbol{d}(\sigma,\tau) + \alpha_1(1-\alpha_2)\boldsymbol{d}(\sigma,\tau+1) \\ &\quad +(1-\alpha_1)\alpha_2\boldsymbol{d}(\sigma+1,\tau) \\ &\quad +(1-\alpha_1)(1-\alpha_2)\boldsymbol{d}(\sigma+1,\tau+1) \quad (7a) \\ \dot{x} &= \alpha_1\alpha_2\boldsymbol{h}(\sigma,\tau) + \alpha_1(1-\alpha_2)\boldsymbol{h}(\sigma,\tau+1) \\ &\quad +(1-\alpha_1)\alpha_2\boldsymbol{h}(\sigma+1,\tau) \\ &\quad +(1-\alpha_1)(1-\alpha_2)\boldsymbol{h}(\sigma+1,\tau+1) \quad (7b) \end{aligned}$$

where

$$\begin{aligned} \alpha_1(x_1) &= \frac{d_1(\sigma+1) - x_1}{d_1(\sigma+1) - d_1(\sigma)}, \quad 0 \le \alpha_1 \le 1 \quad (8a) \\ \alpha_2(x_2) &= \frac{d_2(\tau+1) - x_2}{d_2(\tau+1) - d_2(\tau)}, \quad 0 \le \alpha_2 \le 1. \quad (8b) \end{aligned}$$

We note that a parametric expresion implies a singleton-rule expression at each vertex :

$$\boldsymbol{x} = \boldsymbol{d}(i,j) \longmapsto \dot{x} = \boldsymbol{h}(i,j) \quad (9)$$

where $i = \sigma, \sigma+1$, $j = \tau, \tau+1$.

Define $\alpha_1^o \equiv \alpha_1(0)$, $\alpha_2^o \equiv \alpha_2(0)$ and for $i = 1, 2$

$$\begin{aligned} a_{i1} &= \frac{h_i(\sigma+1,\tau) - h_i(\sigma,\tau)}{d_1(\sigma+1) - d_1(\sigma)} \quad (10a) \\ a_{i2} &= \frac{h_i(\sigma,\tau+1) - h_i(\sigma,\tau)}{d_2(\tau+1) - d_2(\tau)} \quad (10b) \\ a_{i1}^+ &= \frac{h_i(\sigma+1,\tau+1) - h_i(\sigma,\tau+1)}{d_1(\sigma+1) - d_1(\sigma)} \quad (11a) \\ a_{i2}^+ &= \frac{h_i(\sigma+1,\tau+1) - h_i(\sigma+1,\tau)}{d_2(\tau+1) - d_2(\tau)} \quad (11b) \end{aligned}$$

Then we also obtain a state-space expression for a two-dimensional system.

State-space expression[5] *The fuzzy system (5) has the following expression :*

$$\begin{aligned} \dot{x} &= A_{\sigma\tau}(x)\boldsymbol{x} + \boldsymbol{\mu}_{\sigma\tau}, \quad x \in R_{\sigma\tau} \quad (12) \\ \boldsymbol{\mu}_{\sigma\tau} &= \alpha_1^o\alpha_2^o\boldsymbol{h}(\sigma,\tau) + \alpha_1^o(1-\alpha_2^o)\boldsymbol{h}(\sigma,\tau+1) \\ &\quad +(1-\alpha_1^o)\alpha_2^o\boldsymbol{h}(\sigma+1,\tau) \\ &\quad +(1-\alpha_1^o)(1-\alpha_2^o)\boldsymbol{h}(\sigma+1,\tau+1) \quad (13) \end{aligned}$$

where for $\boldsymbol{x} \in R_{\sigma\tau}^o$, $\boldsymbol{\mu}_{\sigma\tau} = \boldsymbol{0}$. Though the matrix $A_{\sigma\tau}$ has four equivalent expression, hereafter it will be expressed as follow :

$$A_{\sigma\tau} \;:\; \alpha_2 S(\tau) + (1-\alpha_2)S(\tau+1) \quad (14)$$

$$S(\tau) = \begin{pmatrix} a_{11} & \alpha_1^o a_{12} + (1-\alpha_1^o)a_{12}^+ \\ a_{21} & \alpha_1^o a_{22} + (1-\alpha_1^o)a_{22}^+ \end{pmatrix} \quad (15a)$$

$$S(\tau+1) = \begin{pmatrix} a_{11}^+ & \alpha_1^o a_{12} + (1-\alpha_1^o)a_{12}^+ \\ a_{21}^+ & \alpha_1^o a_{22} + (1-\alpha_1^o)a_{22}^+ \end{pmatrix} \quad (15b)$$

From (14) we know that the fuzzy system in (12) is characterized as a piecewise-polytopic-affine system where $\dot{x} = S(\tau)x + \boldsymbol{\mu}_{\sigma\tau}$ and $\dot{x} = S(\tau+1)\boldsymbol{x} + \boldsymbol{\mu}_{\sigma\tau}$ are called extreme systems.

From the above state-space expression the vertex condition of the system is expressed as

$$VC \;:\; \boldsymbol{h}(i,j) = A_{\sigma\tau}(\boldsymbol{d}(i,j))\boldsymbol{d}(i,j) + \boldsymbol{\mu}_{\sigma\tau} \quad (16)$$

where $i = \sigma, \sigma+1$, $j = \tau, \tau+1$. Moreover, from this condition we can obtain the singleton-rules (9) and the parametric expression (7). Thus we can derive a state-space expression from a parametric expression and vice versa.

We put a zero-equalibrium condition ZC to the state-space expression which states that $\boldsymbol{x} = \boldsymbol{0} \mapsto \dot{x} = \boldsymbol{0}$.

$$ZC \;:\; \boldsymbol{\mu}_{\sigma\tau} = \boldsymbol{0}, \quad \boldsymbol{x} \in R_{\sigma\tau}^o. \quad (17)$$

Now we consider the stability of the fuzzy system (5). Define a Lyapunov function by $V(\boldsymbol{x}) = \boldsymbol{x}^T P\boldsymbol{x}$, $P > 0$. Then the derivative of $V(\boldsymbol{x})$ is obtained as $\dot{V}(\boldsymbol{x}) = \dot{x}^T P\boldsymbol{x} + \boldsymbol{x}^T P\dot{x}$. We consider $\dot{V}(\boldsymbol{x})$ in a region $R_{\sigma\tau}$. From (7) we can derive two expressions :

$$\begin{aligned} \dot{V}(x) &= \alpha_1\dot{V}(\sigma,*) + (1-\alpha_1)\dot{V}(\sigma+1,*) \\ &\quad -\alpha_1(1-\alpha_1)E(\cdot,*) \quad (18a) \\ \dot{V}(x) &= \alpha_2\dot{V}(*,\tau) + (1-\alpha_2)\dot{V}(*,\tau+1) \\ &\quad -\alpha_2(1-\alpha_2)E(*,\cdot) \quad (18b) \end{aligned}$$

where

$$\begin{aligned} \dot{V}(i,*) &= 2\boldsymbol{h}(i,*)^T P\boldsymbol{d}(i,*) \quad (19a) \\ \dot{V}(*,j) &= 2\boldsymbol{h}(*,j)^T P\boldsymbol{d}(*,j) \quad (19b) \\ i &= \sigma,\sigma+1, \; j = \tau,\tau+1 \end{aligned}$$

where $\boldsymbol{h}(i,*)$ is the value of $\dot{x}$ at $\boldsymbol{x} = \boldsymbol{d}(i,*)$, and $\boldsymbol{d}(i,*) = (d_1(i), d_2(*))^T$, $d_2(\tau) \le d_2(*) \le d_2(\tau+1)$. In a similar manner, $\boldsymbol{h}(*,j)$ and $\boldsymbol{d}(*,j)$ are defined.

$$\begin{aligned} &E(\cdot,*) \quad (20a) \\ &= 2(\boldsymbol{h}(\sigma+1,*) - \boldsymbol{h}(\sigma,*))^T P(\boldsymbol{d}(\sigma+1,*) - \boldsymbol{d}(\sigma,*)) \\ &E(*,\cdot) \quad (20b) \\ &= 2(\boldsymbol{h}(*,\tau+1) - \boldsymbol{h}(*,\tau))^T P(\boldsymbol{d}(*,\tau+1) - \boldsymbol{d}(*,\tau)) \end{aligned}$$

We call the following inequalities concerning above expressions stable vertex conditions SVC and stable edge conditions SEC, respectively,

$$SVC \;:\; \dot{V}(i,j) < 0 \quad (21)$$

$$\begin{aligned} SEC \;:\; E(i,\cdot) &> -\left(\sqrt{-\dot{V}(i,j)} + \sqrt{-\dot{V}(i,j+1)}\right)^2 \\ E(\cdot,j) &> -\left(\sqrt{-\dot{V}(i,j)} + \sqrt{-\dot{V}(i+1,j)}\right)^2 \quad (22) \end{aligned}$$

where $i = \sigma, \sigma+1$, $j = \tau, \tau+1$.

With the above preparations, we have the following stability theorem.

Theorem 1 (Stability Theorem [5],[7]) *Consider a piecewise-polytopic-affine system such that*

$$\begin{aligned} \boldsymbol{x}(t) &= A(\alpha_2)\boldsymbol{x}(t) + \boldsymbol{\mu}_{\sigma\tau}, \quad \boldsymbol{x}(t) \in R_{\sigma\tau} \\ A(\alpha_2) &= \alpha_2 S(\tau) + (1-\alpha_2)S(\tau+1). \end{aligned}$$

where $\boldsymbol{\mu}_{\sigma\tau} = \mathbf{0}$ *in* $R^o_{\sigma\tau}$.
The system is asymptotically stable in the large if there exists a common $P > 0$ *such that*

(1) in the zero-square, SVC, SEC and SZC are satisfied, where $SZC: -A(\alpha_2^o)^T P - PA(\alpha_2^o) > 0$,

(2) in the other regions, SVC and modified SEC are satisfied, where SEC :

$$\begin{aligned} E(i,\cdot) \geq & -\Bigg(\sqrt{-\gamma(i,j)\dot{V}(i,j)} \\ & + \sqrt{-\gamma(i,j+1)\dot{V}(i,j+1)}\Bigg)^2 \\ E(\cdot,j) \geq & -\Bigg(\sqrt{-(1-\gamma(i,j))\dot{V}(i,j)} \\ & + \sqrt{-(1-\gamma(i+1,j))\dot{V}(i+1,j)}\Bigg)^2 \end{aligned}$$

where

$$0 < \gamma(i,j) < 1, \quad i = \sigma, \sigma+1, \; j = \tau, \tau+1.$$

We assume that the equalities do not hold at the same time in the above inequalities.

3. Stabilizing Control

In this section, we consider a state-feedback stabilizing control of the type II fuzzy systems and present a stabilization algorithm for the model-based design of the type II fuzzy controllers.

In the sequel we shall restrict the object of control to the following (non)linear system :

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}u \tag{23}$$

where $\boldsymbol{f}(\boldsymbol{x})$ is a fuction vector, $\boldsymbol{b}$ is a constant vector and u is a scalar input.

Consider a state-feedback fuzzy controller of type II

$$if\ \boldsymbol{x}\ is\ G^{\sigma\tau}(\boldsymbol{x}),\ then\ u\ is\ l(\sigma,\tau). \tag{24}$$

We obtain a parametric expression of the above fuzzy controller

$$\begin{aligned} u(t) = & \alpha_1\alpha_2 l(\sigma,\tau) + \alpha_1(1-\alpha_2)l(\sigma,\tau+1) \\ & +(1-\alpha_1)\alpha_2\, l(\sigma+1,\tau) \\ & +(1-\alpha_1)(1-\alpha_2)l(\sigma+1,\tau+1). \end{aligned} \tag{25}$$

Define

$$a_1 = \frac{l(\sigma+1,\tau) - l(\sigma,\tau)}{d_1(\sigma+1) - d_1(\sigma)} \tag{26a}$$

$$a_2 = \frac{l(\sigma,\tau+1) - l(\sigma,\tau)}{d_2(\tau+1) - d_2(\tau)} \tag{26b}$$

$$a_1^+ = \frac{l(\sigma+1,\tau+1) - l(\sigma,\tau+1)}{d_1(\sigma+1) - d_1(\sigma)} \tag{27a}$$

$$a_2^+ = \frac{l(\sigma+1,\tau+1) - l(\sigma+1,\tau)}{d_2(\tau+1) - d_2(\tau)}. \tag{27b}$$

Then we also obtain a state-space expression

$$u(t) = \boldsymbol{c}_{\sigma\tau}^T \boldsymbol{x} + \xi_{\sigma\tau}, \quad \boldsymbol{x} \in R_{\sigma\tau} \tag{28}$$

$$\begin{aligned} \xi_{\sigma\tau} = & \alpha_1^o\alpha_2^o l(\sigma,\tau) + \alpha_1^o(1-\alpha_2^o)l(\sigma,\tau+1) \\ & +(1-\alpha_1^o)\alpha_2^o l(\sigma+1,\tau) \\ & +(1-\alpha_1^o)(1-\alpha_2^o)l(\sigma+1,\tau+1) \end{aligned} \tag{29}$$

where for $\boldsymbol{x} \in R^o_{\sigma\tau}$ $\xi_{\sigma\tau} = 0$. Though the vector $\boldsymbol{c}_{\sigma\tau}$ has two equivalent expressions, we shall use the following expression coresssponding to (14) and (15).

$$\boldsymbol{c}_{\sigma\tau} \;:\; \alpha_2\boldsymbol{k}(\tau) + (1-\alpha_2)\boldsymbol{k}(\tau+1) \tag{30}$$

$$\boldsymbol{k}(\tau) = (a_1\ ,\ \alpha_2^o a_2 + (1-\alpha_2^o)a_2^+)^T \tag{31a}$$

$$\boldsymbol{k}(\tau+1) = (a_1^+\ ,\ \alpha_2^o a_2 + (1-\alpha_2^o)a_2^+)^T. \tag{31b}$$

Applying the fuzzy control (24) to the fuzzy system (1), we obtain a closed-loop system

$$if\ \boldsymbol{x}\ is\ G^{\sigma\tau}(\boldsymbol{x}),\ then\ \dot{\boldsymbol{x}}\ is\ \boldsymbol{h}(\sigma,\tau) + \boldsymbol{b}l(\sigma,\tau) \tag{32}$$

where the singleton consequent in (32) means the value of $\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}u$ in (23) for $\boldsymbol{x} = \boldsymbol{d}(\sigma,\tau)$.
From (7) and (25), we can derive a parametric expression of the closed-loop system (32)

$$\begin{aligned} \dot{\boldsymbol{x}} = & \alpha_1\alpha_2\hat{\boldsymbol{h}}(\sigma,\tau) \\ & +\alpha_1(1-\alpha_2)\hat{\boldsymbol{h}}(\sigma,\tau+1) \\ & +(1-\alpha_1)\alpha_2\hat{\boldsymbol{h}}(\sigma+1,\tau) \\ & +(1-\alpha_1)(1-\alpha_2)\hat{\boldsymbol{h}}(\sigma+1,\tau+1) \end{aligned} \tag{33}$$

where

$$\hat{\boldsymbol{h}}(i,j) = \boldsymbol{h}(i,j) + \boldsymbol{b}l(i,j). \tag{34}$$

And also, from (12) and (28) we can derive a state-space expression of the closed-loop system

$$\dot{\boldsymbol{x}} = \hat{A}_{\sigma\tau}\boldsymbol{x} + \hat{\boldsymbol{\mu}}_{\sigma\tau} \tag{35}$$

$$\begin{aligned} \hat{\boldsymbol{\mu}}_{\sigma\tau} = & \alpha_1^o\alpha_2^o\hat{\boldsymbol{h}}(\sigma,\tau) + \alpha_1^o(1-\alpha_2^o)\hat{\boldsymbol{h}}(\sigma,\tau+1) \\ & +(1-\alpha_1^o)\alpha_2^o\hat{\boldsymbol{h}}(\sigma+1,\tau) \\ & +(1-\alpha_1^o)(1-\alpha_2^o)\hat{\boldsymbol{h}}(\sigma+1,\tau+1) \end{aligned} \tag{36}$$

where

$$\hat{A}_{\sigma\tau} = A_{\sigma\tau} + \boldsymbol{b}\boldsymbol{c}_{\sigma\tau}^T, \tag{37a}$$

$$\hat{\boldsymbol{\mu}}_{\sigma\tau} = \boldsymbol{\mu}_{\sigma\tau} + \boldsymbol{b}\xi_{\sigma\tau}. \tag{37b}$$

It is seen that the closed-loop systems (33) and (35) are of the same forms as (7) and (14), respectively. Therefore, it is possible to apply Theorem 1 for the feedback control system. Our idea for stabilizing control is to assign vertices by adjusting the singleton consequents of a control law so that the closed-loop system satisfies the stability conditions. We assume that for all regions, extreme affine systems are controllable in order to guarantee the vertex-assignment[6].

Now we discuss a stabilization algorithm.

We consider the problem of finding a feedback control $u(\boldsymbol{x})$ for the fuzzy system (32) with the following properties :

(i) it achieves asymptotic stability of the equilibrium $\boldsymbol{x} = \boldsymbol{0}$,

(ii) it minimizes the cost function

$$J = \int_0^\infty (Q(\boldsymbol{x}) + R(u))\, dt \qquad (38)$$

where $Q(\boldsymbol{x}) > 0$, $R(u) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$, $u \neq 0$.

Generally it is not a simple task to solve the problem for a fuzzy system or a nonlinear system. Here we will take an inverse-problem-approach of optimal control. For a certain $P > 0$, we first assume an optimal control law at each vertex as

$$l(i,j) = -\frac{1}{r(i,j)}\boldsymbol{b}^T P \boldsymbol{d}(i,j), \qquad (39)$$
$$r(i,j) > 0,\ P > 0,\ i = \sigma, \sigma+1,\ j = \tau, \tau+1.$$

For an unforced system where $l(i,j) = 0$ in (39), the derivative of $V(\boldsymbol{x}) = \boldsymbol{x}^T P \boldsymbol{x}$ is obtained as

$$\begin{aligned} \dot{V}_o = {} & 2[\alpha_1\alpha_2 \boldsymbol{h}(\sigma,\tau) + \alpha_1(1-\alpha_2)\boldsymbol{h}(\sigma,\tau+1) \\ & + (1-\alpha_1)\alpha_2 \boldsymbol{h}(\sigma+1,\tau) \\ & + (1-\alpha_1)(1-\alpha_2)\boldsymbol{h}(\sigma+1,\tau+1)]^T P \\ & [\alpha_1\alpha_2 \boldsymbol{d}(\sigma,\tau) + \alpha_1(1-\alpha_2)\boldsymbol{d}(\sigma,\tau+1) \\ & + (1-\alpha_1)\alpha_2 \boldsymbol{d}(\sigma+1,\tau) \\ & + (1-\alpha_1)(1-\alpha_2)\boldsymbol{d}(\sigma+1,\tau+1)]. \end{aligned} \qquad (40)$$

Denote $\dot{V}_o$ at each vertex as $\dot{V}_o(i,j)$. We have

$$\dot{V}_o(i,j) = 2\boldsymbol{h}(i,j)^T P \boldsymbol{d}(i,j), \qquad (41)$$
$$i = \sigma, \sigma+1,\ j = \tau, \tau+1.$$

Set $R(u) = -u(\boldsymbol{x})\boldsymbol{b}^T P \boldsymbol{x}$ and denote $R(u)$ at each vertex as $R(i,j)$. $R(i,j)$ is expressed as

$$\begin{aligned} R(i,j) &= -l(i,j)\boldsymbol{b}^T P \boldsymbol{d}(i,j) \qquad (42) \\ &= r(i,j)\ l(i,j)^2 \\ &= \frac{(\boldsymbol{b}^T P \boldsymbol{d}(i,j))^2}{r(i,j)} \end{aligned}$$

where $r(i,j) > 0$, $i = \sigma, \sigma+1$ and $j = \tau, \tau+1$. Now $\dot{V}$ for a control input $\frac{1}{2}u(\boldsymbol{x})$ is expressed as

$$\dot{V}_{\frac{1}{2}u} = \dot{V}_o - R(u). \qquad (43)$$

Setting $Q(\boldsymbol{x}) = -\dot{V}_{\frac{1}{2}u}$, we obtain

$$\begin{aligned} \dot{V} &= -Q(\boldsymbol{x}) - R(u) \\ &= \dot{V}_o - 2R(u). \end{aligned} \qquad (44)$$

Therefore, if $R(u) > 0$ and $\dot{V}_{\frac{1}{2}u} < 0$, i.e., $Q(\boldsymbol{x}) > 0$, we can say that u composed of (39) is an optimal control law.

Hence, the strategy of a stabilization alogorithm is as follows : We first set an appropriate J. Then we assign vertices so that (i) $R(i,j) > 0$ and (ii) $\dot{V} < 0$ for $\forall x$ in the inside of a region ; as for $R(u)$, we do not require that $R(u) > 0$ in the inside of a region.

[Stabilization algorithm]

(step 1) Check the controllability of the system.

In each region we check the controllability of extreme systems in order to guarantee the vertex-assignment.

(step 2) Set a base system and parameters.

In the zero-square by setting $\alpha_2 = \alpha_2^o (\equiv \alpha_2(0))$ we have

$$\begin{aligned} \dot{\boldsymbol{x}} &= S_o\boldsymbol{x} + \boldsymbol{b}u, \quad \boldsymbol{x} \in R^o_{\sigma\tau} & (45) \\ S_o &= \alpha_2^o S(\tau) + (1-\alpha_2^o)S(\tau+1). & (46) \end{aligned}$$

We use (45) as a base system for all regions.

Then we assume that $\boldsymbol{x}^T Q_o \boldsymbol{x} + r_o u^2$ where $Q_o > 0$, $r_o > 0$ and set an optimal u as $u_z = -\frac{1}{r_o}\boldsymbol{b}P_o\boldsymbol{x}$. As usual P_o is obtained by a Ricatti equation. We use this P_o as a common P for all regions. We have for u_z

$$\dot{V}_z = -\boldsymbol{x}^T Q_o \boldsymbol{x} - r_o u_z^2 \qquad (47)$$

and at vertices

$$\begin{aligned} \dot{V}_z(i,j) &= -\boldsymbol{d}(i,j)^T Q_o \boldsymbol{d}(i,j) - r_o u_z(i,j)^2 & (48) \\ u_z(i,j) &= -\frac{1}{r_o}\boldsymbol{b}^T P_o \boldsymbol{d}(i,j). & (49) \end{aligned}$$

We set a range of control input as $u_r \geq |u| > 0$, which will be used in (step 5).

(step 3) Determine a control law at each vertex.

At each vertex we determine the parameters $r(i,j)$ in (39) such that

$$\dot{V}(i,j) \leq \dot{V}_z(i,j)\ , \quad \forall i,j\ . \qquad (50)$$

We introduce a parameter c in order to bring about a damping effect. We consider two cases.

(i) For the case that $\dot{V}_o(i,j) > \dot{V}_z(i,j)$.

Denote the maximum of feasible $r(i,j)$ by r_s where r_s is obtained as

$$r_s(i,j) = \frac{2(\boldsymbol{b}^T P_o \boldsymbol{d}(i,j))^2}{\dot{V}_o(i,j) - \dot{V}_z(i,j)} > 0. \qquad (51)$$

Applying a parameter c to (51), we set an upper bound r^* as

$$\frac{1}{r^*(i,j)} = \frac{1}{r_s(i,j)} + c, \qquad c > 0 \qquad (52)$$

(ii) For the case that $\dot{V}_o(i,j) \leq \dot{V}_z(i,j)$.

Since a closed-loop system satisfies already (50), we can say that $u = 0$ is a candidate of stabilizing control laws. In this case we try to reduce $\dot{V}(i,j)$

as much as possible by setting an upper bound r^* as follows :

$$\frac{1}{r^*(i,j)} = c, \qquad c > 0 \qquad (53)$$

Finally we determine the values of $r(i,j)$ in the interval $(0, r^*(i,j)]$ given by (52) or (53).

(step 4) Check the stability of a closed-loop system.

Using $r(i,j)$ and a common $P_o > 0$, we check the stability of a closed-loop system in each region. If a system satisfies the stability conditions of Theorem 1, the $l(i,j)$ obtained from $r(i,j)$ gives a stabilizing control law. if not, we set $V_z(i,j) = 0$ and then excute (step 3).

(step 5) Improve a damping effect.

Using $l(i,j)$ obtained from (step 4) we calculate the maximum value of control input denoted by $l_m = \max_{i,\,j} |l(i,j)|$. And we change the parameter c in (52) or (53) and then iterate (step 3) and (step 4) until the maximum value falls in an allowing range, for instance, $0.99u_r \le l_m \le u_r$.

4. Design Example

We design a stabilizing controller for a type II fuzzy model of the well-known Van der Pol system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \epsilon(1 - x_1^2)x_2 + u. \end{aligned} \qquad (54)$$

where $x_1 \in [-2.5 \;\; 2.5]$, $x_2 \in [-3.5 \;\; 3.5]$, $u \in [-15 \;\; 15]$ and $\epsilon = 1$. Table 1 shows the vertex condition of an approximated fuzzy model where the number of regions is 45. And Fig. 2 shows the nonlinearity of system (54).

We illustrate a design process of a type II fuzzy controller.

(step 1) This system is controllable for all regions.

(step 2) In the zero-square S_o and b are

$$S_o = \begin{pmatrix} 0.00 & 1.00 \\ -1.00 & 0.99 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Using the above parameters, a base system is obtained from (45). We choose

$$Q_o = \begin{pmatrix} 10.0 & -3.1 \\ -3.1 & 1.0 \end{pmatrix}, \quad r_o = 1$$

By solving a Ricatti equation, we obtain

$$P_o = \begin{pmatrix} 12.6192 & 2.3166 \\ 2.3166 & 3.5616 \end{pmatrix}.$$

We use this P_o as a common P for all regions. And we calculate $V_z(i,j)$ in (49) which is shown in Fig. 3. From (54) we set a range of control input as $u_r = 15$.

(step 3, 4 and 5) From (52) or (53) we set $c = 2^{-52}$ and calculate $r^*(i,j)$. We initially choose $r(i,j)$ as the upper bound $r^*(i,j)$. And then check the stability of the closed-loop system. In this example the closed-loop system satisfies the stability conditions for all regions. Thus from (step 5) we have $c = 0.1572$. We can verify the stability with the vertex condition in (16) for the values of parameters $\gamma(i,j)$ of *SEC* in Theorem 1 which are shown in Table 2.

The designed control law is shown in Table 3. Fig. 4 shows the input-ouput relation of the fuzzy controller, where we see that the relation is nonlinear.

Fig. 5 shows the variable $\dot{x}_2$ of the closed-loop system on the state-space. We know that the surface of $\dot{x}_2$ is similar to a plane, i.e., linear.

For all regions of the closed-loop system, $\dot{V}$ is shown in Fig. 3 where we see that the inequality (50) almost holds. Fig 6 and Fig 7 show phase potraits of the open-loop system (dotted line) and the closed-loop system (solid line) for the initial states $\boldsymbol{x}(\mathbf{0}) = (2.25,\ 0)^T$ and $\boldsymbol{x}(\mathbf{0}) = (0.25,\ 0)^T$, respectively. The open-loop system has limit cycle. We find that the closed-loop system converges to the origin $(0,0)$ for both inner and outer initial states of the limit cycle.

5. Conclusion

We have discussed the stability of a type II fuzzy system and presented a stabilization algorithm based on an inverse-problem-approach of optimal control. A design example has been shown for the Van der Pol system.

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Table 1 Fuzzy model of Van der Pol system

$h_2(d_1, d_2)$		$d_2(1)$	$d_2(2)$	$d_2(3)$	$d_2(4)$	$d_2(5)$	$d_2(6)$
		-3.5	-1.5	-0.3	0.3	1.5	3.5
$d_1(1)$	-2.5	20.875	10.375	4.075	0.925	-5.375	-15.875
$d_1(2)$	-1.7	8.315	4.535	2.267	1.133	-1.135	-4.915
$d_1(3)$	-1.0	1.000	1.000	1.000	1.000	1.000	1.000
$d_1(4)$	-0.5	-2.125	-0.625	0.275	0.7250	1.6250	3.125
$d_1(5)$	-0.1	-3.365	-1.385	-0.197	0.397	1.585	3.565
$d_1(6)$	0.1	-3.565	-1.585	-0.397	0.197	1.385	3.365
$d_1(7)$	0.5	-3.125	-1.625	-0.725	-0.275	0.625	2.125
$d_1(8)$	1.0	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000
$d_1(9)$	1.7	4.915	1.135	-1.133	-2.267	-4.535	-8.315
$d_1(10)$	2.5	15.875	5.375	-0.925	-4.075	-10.375	-20.875

Table 2 A example of parameter γ

region	$\gamma(\sigma,\tau)$	$\gamma(\sigma,\tau+1)$	$\gamma(\sigma+1,\tau)$	$\gamma(\sigma+1,\tau+1)$
$[d_1(4)\ d_1(5)] \times [d_2(3)\ d_2(4)]$	0.500	0.500	0.969	0.857
$[d_1(6)\ d_1(7)] \times [d_2(3)\ d_2(4)]$	0.857	0.969	0.500	0.500
the other region($x \notin R^{\sigma}_{\tau}$)	0.500	0.500	0.500	0.500

Table 3 Fuzzy controller of Van der Pol

$l(d_1, d_2)$		$d_2(1)$	$d_2(2)$	$d_2(3)$	$d_2(4)$	$d_2(5)$	$d_2(6)$
		-3.5	-1.5	-0.3	0.3	1.5	3.5
$d_1(1)$	-2.5	2.8700	3.5243	6.0665	7.3375	9.8797	-1.0492
$d_1(2)$	-1.7	8.9027	6.4197	4.9298	4.1849	-0.2207	-1.3405
$d_1(3)$	-1.0	13.6412	7.3781	3.6203	1.7413	-2.0165	-8.2796
$d_1(4)$	-0.5	14.9258	7.1627	2.5049	0.1759	-4.4819	-12.2450
$d_1(5)$	-0.1	14.6934	6.4504	1.5045	-0.9684	-5.9142	-14.1573
$d_1(6)$	0.1	14.1573	5.9142	0.9684	-1.5045	-6.4504	-14.6934
$d_1(7)$	0.5	12.2450	4.4819	-0.1759	-2.5049	-7.1627	-14.9258
$d_1(8)$	1.0	8.2796	2.0165	-1.7413	-3.6203	-7.3781	-13.6412
$d_1(9)$	1.7	1.3405	0.2207	-4.1849	-4.9298	-6.4197	-8.9027
$d_1(10)$	2.5	1.0492	-9.8797	-7.3375	-6.0665	-3.5243	-2.8700

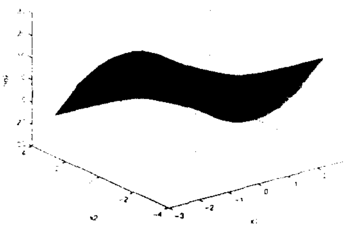

Fig. 2 State-space representation of $\dot{x}_2$ (open-loop system)

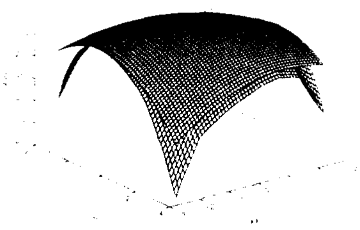

Fig. 3 V_2(above) and $\dot{V}$(bellow)

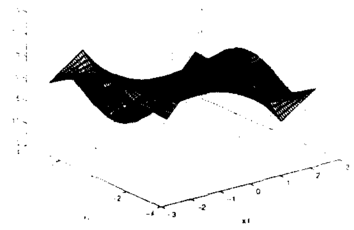

Fig. 4 Fuzzy controller

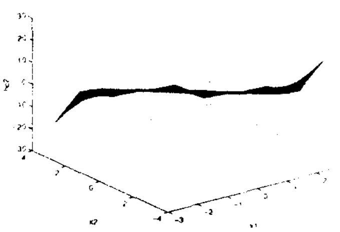

Fig. 5 State-space representation of $\dot{x}_2$ (closed-loop system)

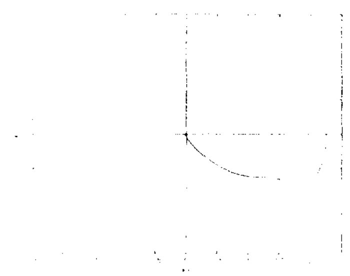

Fig. 6 Phase portrait for $x(0) = (2.25, 0)^T$

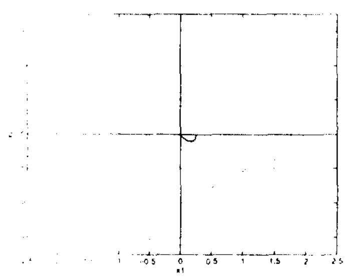

Fig. 7 Phase portrait for $x(0) = (0.25, 0)^T$