The Third AFSS(1998). 1－19

THEORIES OF SET AND LOGIC: COMPUTING WITH WORDS AND NUMBERS

I.B. Tűrkşen[1]
Department of Mechanical and Industrial Engineering
University of Toronto
Toronto, Ontario M5S 3G8 Canada

Phone: (416) 978-1278
Fax: (416) 978-3453
email: tűrkşen@mie.utoronto.ca

ABSTRACT

In this Keynote Address, two types of information granules are considered: (i) one for set assignments of a concept descriptor and (ii) the other for truthood assignment to the concept description verifier. The first is, the process which specifies the assignment of an object to a clump, a class, a group, etc., and hence defines the set membership with a relational constraint. The second is the assignment of the degree of truthood or the membership specification of the abstract concept of truthood which specifies the "veristic" constraint associated with the concept descriptor. The combination of these two distinct assignments let us generate four set and logic theories. This then leads to the concern of normal forms and their derivation from truth tables for each of these theories. In this regard, some of the fundamental issues arising in this context are discussed and certain preliminary answers are provided in order to highlight the consequences of these theories.

[1] Supported in part by Natural Science and Engineering Research Council of Canada, NSERC

1. INTRODUCTION

Computing with words may be implemented by an assignment of numbers either in {0,1} or in [0,1] to linguistic terms of linguistic variables in a natural language. Linguistic terms are special words that are labels for information granules that are represented by sets in formal theories. Therefore, every information granule has two assignments: (i) a word that summaries our knowledge in a natural language, and (ii) a set of numbers that represents the word in a computational theory.

Thus, we develop a computational apparatus by representing our knowledge on the one hand with words and on the other hand with numbers. In two-valued paradigm, the words are assigned to {0,1}, but in infinite (fuzzy)-valued paradigm, the words are assigned to [0,1]. Next, it should be recalled that there are many different classes of words that summarized our knowledge. The two unique classes of words that we concentrate on here are (i) the words that describe our knowledge for which we use descriptive sets assignments and (ii) the words that verify the descriptive set assignments specified in (i) above for which we use verification set assignments.

Thus there are in effect two distinct but related assignments. It appears that these two distinct assignments are implicitly assumed to be the same in most articles of the current fuzzy theory literature. It further appears that this confusion arises because both of the assignments are values either in {0,1} for the two valued theory or in [0,1] for the infinite valued theory. However, it is worthwhile to re-emphasize that one specifies an object's degree of belonging to a set identified by a *label, a word of a concept category*, and the other specifies the degree of belonging of the set assignment given in the first instant to our abstract concept of *verity, i.e., truthood.*

As a mater-of-fact, it is common knowledge that there are two distinct disciplines in the institutions of higher learning: (1) "Set" theory and (2) "Logic" theory. Furthermore, it is to be recalled that logicians start out their discussions usually with an expression which state: "it is the case that (. . .), i.e., $\models$ (. . .), where (. . .) represents a proposition, a set definition, which is verified, asserted or denied, and where the symbol "$\models$" is a short hand notation for "it is the case that". Many other examples of such expressions may be found in introductory logic books [1,5]; e.g., "it is true that (. . .)", "it is affirmed that (. . .)", etc. Introductory logic books make this point rather clear at the outset.

Therefore, in fuzzy set and logic research, one needs to come to realize that these two concepts of "sethood" and "truthood" must be given due recognition in the fundamental derivations of the theoretical constructs such as truth tables, well formed formulas, normal forms, etc.

In this context, the central concern of this discourse is directed to the formation of generalized truth tables and normal form expressions. It is hoped that this will shed further light on the future discussions while we are reassessing the foundations of fuzzy information granulation toward a well founded theory.

In order to add further clarity to this issue of concern, the following two definitions are needed for the discourse to be build on a well defined foundation.

Definition 1. Set Assignment:

Every element (object, person) x belongs to a set X with the set membership assignment, $\mu_A(x)$ with respect to a fuzzy information granule, a linguistic term, A, of a linguistic variable, $\mathcal{A}$, which is generally a word in a natural language or a label of a concept category.

Let $s \in S_D$ be this assignment statement (Definition 1) given above in the set of all possible statements S_D which denotes Set "Description", D.

Then symbolically, the set membership assignment may be expressed as:

$$s \ ::= x \in X \text{ isr } A,\ \mu_A(x) \tag{1}$$

where "isr" [31] indicates that the membership assignment, $\mu_A(x) \in [0,1]$, for $x \in X$, is in relation to a fuzzy information granule A of $\mathcal{A}$ such that $\mu_A: X \to [0,1]$. In two-valued theory, we have:

$$s \ ::= x \in X \text{ is } A,\ \mu_A(x)$$

where "is" indicates that the set membership assignment, $\mu_A(x) \in \{0,1\}$ for $x \in X$, is in a crisp information granule A of $\mathcal{A}$ such that $\mu_A: X \to \{0,1\}$.

Definition 2. Verity Assignment:

Every set assignment s , S_D specified by Definition 1 above, has an associated truthood assignment $\mu_T(s)$ with respect to the concept of verification of s which may be affirmed or denied to a degree by a person (a sensor) and that such a verification may be uttered, specified or determined, either by the same person (sensor) or an independent person (sensor) that is other than the person (sensor) that makes the set assignment s , S_D in the first place.

Let "$\models$ " stand for "it is the case that" as the usually accepted symbol in logic. Then the verity assignment statement t , S_V within the set of veristic "Truthood" statements is defined as:

$$t \;::'\; s,\ S_D \text{ isv } T, \mu_T(s) \text{or} \tag{2}$$

$$t \;::'\; (x,\ X \text{ isr } A, \mu_A(x))\ ,\ S_D \text{ isv } T, \mu_T(s), \tag{3}$$

where t , S_V, a truth verification statement in the set of truth verification statements S_V, "s :: ' x, X isr $A, \mu_A(x)$ " as defined by (1) above; and "isv" [31] indicates the verity assignment $\mu_T(s)$ for every s , S_D in relation to an information granule of truthood T, of the abstract concept of truthood T, such that T stands for a linguistic term such as "true", "false", "very true", "somewhat true", etc., of the linguistic variable of "Truthood", T, in infinite valued logic theory such that $\mu_T : S_D$ 6 [0, 1], where T , T = {true, false, very true, somewhat true,..., false, very false} or it stands for the dichotomous truthood in two valued logic theory such that, $\mu_T : S_D$ 6 {0, 1} where "1" stands for "true" and "0" stands for "false", i.e., $\mu_T : S_D$ 6 {t}, t , {0, 1}. At times, due to classical convention, we use t , {F, T}.

In particular, it should be re-emphasized that in expression (1), "X isr A" is a fuzzy proposition identified with a fuzzy information granule A, such that "s :: = x , X isr A with $\mu_A(x)$, [0,1]" identifies a single generic element x of X in the fuzzy information granule A with a specific membership value $\mu_A(x)$. On the other hand in expression (2), "$\models s$, S_D isv T, $\mu_T(s)$, [0,1]" is a fuzzy predicate which is truth qualified by $\mu_T(s)$, where "s : : = x , X isr A, $\mu_A(x)$, [0,1]" is a generic but specific element s , S_D, a linguistic statement, more generally identified with the fuzzy proposition

"X isr A". In Figure 1, these concepts are depicted in a *Fuzzy Ven Diagram.*

Furthermore, it should be noted in this context that our knowledge representation, K, is a tuple made up with Set "Description", D, and Truthood "Verification", V, i.e., K = (D, V) is expressed more explicitly as

D: "s :: = x , X isr A, $\mu_A(x)$" for every s , S_D, and

V: "t :: = s , S_D isv T, $\mu_T(s)$", for every t , S_V

Thus it is proposed that the usual notation (x, $\mu_A(x)$), or $\mu_A(x)/x$, be extended to be [(x, $\mu_A(x)$), [s, $\mu_T(s)$] or [$\mu_A(x)/x$, $\mu_T(s)/s$] and to be read "every element x , X is assigned a set membership in an information granule A of A with the membership $\mu_A(x)$, within the context of s :: = x , X isr A, $\mu_A(x)$; such that s , S_D; and it is independently verified by the assignment of truth degree in an information granule T of T with the membership $\mu_T(s)$, within the context of "t :: = s , S_D isv T, $\mu_T(s)$, such that t 0 S_V".

It is to be recalled that a witness, in a *Court of Law*, is asked to affirm or deny a statement, an event, a description, etc., in terms of two-valued logic as "true" or "false" which is separate from the statement that describes a situation and in addition to such a statement.

In this regard, let us next briefly review the interpretations of membership concepts from the current literature.

2. SET AND LOGIC COMBINATION

In the light of the foregoing discussion given above, it is natural to propose four basic theories and one myopic theory that combine the set and logic assignments that form the basis of our formal knowledge representation. These are: (i) the set and logic theory based on two-valued set and two-valued logic assignments. This is the well known Boolean set and logic theory. Let us call this $S_{\{0,1\}}$ $L_{\{0,1\}}$ theory; (ii) the set and logic theory based on infinite valued (fuzzy) set and two valued logic assignments. Let us call this $S_{[0,1]}$ $L_{\{0,1\}}$ theory; (iii) the set and logic theory based on two-valued set but infinite valued (fuzzy) logic assignments. Let us call this $S_{\{0,1\}}$ $L_{[0,1]}$ theory; and last (iv) the set and logic theory based on infinite valued (fuzzy) set and infinite valued (fuzzy) logic assignments.

Let us call this $S_{[0,1]} L_{[0,1]}$ theory. The last two theories, i.e., $S_{\{0,1\}} L_{[0,1]}$ and $S_{[0,1]} L_{[0,1]}$ theories, has not been investigated properly to this date.

In our view, these are the four basic theories, but there is a fifth theory which is the one most commonly referred to as the "Fuzzy Logic" theory in the current literature. This is essentially the first theory, i.e., two-valued set and two-valued logic theory that is *fuzzified directly* in a limited manner. Let us call it the *myopic theory* and identify it as $F(S_{\{0,1\}} L_{\{0,1\}})$. In this myopic theory, only the shorter forms of the well formed formulas of $S_{\{0,1\}} L_{\{0,1\}}$ theory are *just fuzzified.* In the myopic theory, the short forms of the well formed formulas are taken at times to be the Disjunctive Normal Form and at other times they are taken to be the Conjunctive Normal Form, i.e., DNF and CNF, respectively, depending on the combination of the chosen concepts. This myopic selection will be further discussed below.

In all these five set and logic theories, combination of any two concept, say A and B, lead to the formation of sixteen well known meta-linguistic concepts as shown in Table 1.

The essential claim of this Keynote Address is that canonical expressions, known as Disjunctive Normal Form and Conjunctive Normal Form, DNF and CNF, respectively, needs to be derived and interpreted in agreement with the principles and axioms related to each of these basic theories. It is further claimed that the derivation of the canonical expressions for $S_{[0,1]} L_{\{0,1\}}$ are a generalization of $S_{\{0,1\}} L_{\{0,1\}}$. Furthermore, it is claimed that Fuzzy Disjunctive and Fuzzy Conjunctive Normal forms, FDNF and FCNF, respectively, are no longer equivalent to each other while it is very well known that DNF (.) ≡ CNF (.) in $S_{\{0,1\}} L_{\{0,1\}}$, for all of the sixteen combined concepts shown in Table 1, where (.) stands for any of the sixteen possible concept combinations.

We now return to each of these four theories in detail to layout down the technical formal ground for this claim.

2.1 $S_{\{0,1\}} L_{\{0,1\}}$ Theory

This is the well known, classical, two-valued set and two-valued logic theory. It is very well known that the DNF and CNF formulas of this theory, as shown in Table 2, are derived from Truth Tables for each of the sixteen meta-linguistic expressions shown in Table 1.

For example, for the meta-linguistic expression #3, i.e., "A OR B", we have, from Table 2 row 3, DNF (A OR B) = (A 1B) c (A 1 c(B)) c (c(A) 1 B), and CNF (A OR B) = A c B, where "1" stand for set conjunction, "c" for set disjunction and c(.) for set complementation.

For the meta-linguistic expression #6, i.e., "A AND B", we have, from Table 2 row 6, DNF (A AND B) = A 1 B, and CNF (A AND B) = (A c B) 1 (A c c(B)) 1 (c(A) c B).

And for the meta-linguistic expression #7, i.e., "A IMPLIES B" or "A 6 B" we have, from Table 2 row 7, DNF (A 6 B) = (A 1 B) c (c(A) 1 B) c (c(A) 1 c(B)), and CNF (A 6 B) = c(A) c B.

Now, in the classical theory DNF (.) / CNF (.) for all the sixteen combinations shown in Table 1 and Table 2, due in part to the axiom of idempotency and in part to the law of Excluded Middle, LEM, and the Law of Crisp Contradiction, LCC, because the set membership assignments $\mu_A(x)$, {0,1} and $\mu_B(y)$, {0,1} for all x , X and y , Y.

2.2 Myopic, F($S_{\{0,1\}}$ $L_{\{0,1\}}$) Theory

In the myopic, F ($S_{\{0,1\}}$ $L_{\{0,1\}}$) theory, researchers usually take "A 1 B" for "A AND B" which is the DNF (A AND B) and fuzzify the set membership values and disregard CNF (A AND B).

In a analogous manner, researchers usually take A c B for "A OR B" which is the CNF (A OR B) and fuzzify the set membership values and disregard DNF (A OR B). However, it was shown [20] that when set membership assignments are taken to be $\mu_A(x)$, [0, 1] and $\mu_B(y)$, [0, 1], DNF(.) ... CNF (.) for all sixteen combinations shown in Table 2, simply because the LEM and LCC do not hold in the crisp sense [28].

In particular, it was shown that DNF(.) f CNF(.) for all the sixteen combination of concepts when the connectives are assumed to be the (w, v, -), i.e., (Max, Min, St Comp), De Morgan Triple [20]. In 1986 and later in several articles [21, 22, 23, 24, 25], it was questioned why many researchers of fuzzy theory insist on using just the CNF of the expression #3, Row 3 of Table 2, i.e., $A \cup B$. But the same researchers insist on using just the DNF of expression #6, Row 6 of Table 2, i.e., $A \cap B$, and the CNF of the expression #7, Row 7 of Table 2, i.e., $c(A) \cup B$.

To repeat the question in a different manner, one should ask why are the researchers ignoring the DNF (A OR B), CNF (A AND B), DNF ($A \rightarrow B$)?

An answer, is that, most of us, having been educated and trained in two-valued set and two-valued logic theory, have become accustomed to using the short form of these expressions since in classical theory DNF(.) ≡ CNF(.)

Hence due to our human habits, most of us did not question what would happen to these canonical forms in just fuzzifying the classical expressions. Thus the myopic fuzzy set and logic theory F ($S_{\{0,1\}}\, L_{\{0,1\}}$) has been in a common usage in most of our current literature. This myopic theory is the one that is identified as the fourth theory. In order to go to the second theory, $S_{[0,1]}\, L_{\{0,1\}}$, one needs to ask the fundamental question: "How are the normal form expression, DNF and CNF are derived in the first place at the very beginning of a set and logic theory and how are they to be restructured in fuzzy set and logic theories?"

One approach is to look at the construction of Truth Tables and the derivation of canonical expressions from the Truth Tables based on the "Normal, Canonical, Form Derivation Algorithm", Appendix I.

2.3 $S_{[0,1]}\, L_{\{0,1\}}$ Theory

In order to develop this theory and its normal forms which are called Fuzzy Disjunctive and Fuzzy Conjunctive Normal Forms, FDNF and FCNF, respectively, one must ask two related questions: (i) whether or not "Fuzzy Truth Tables" can be constructed, and if so (ii) can one derive the canonical, normal, forms, i.e., FDNF and FCNF, from such tables?

These two questions were answered in the affirmative manner in Tűrkşen [21] and later treated from two different unique perspectives in Tűrkşen, *et al.* [27, 28]. Various investigations attempted to show the impact of these FDNF and FCNF in fuzzy system modeling [22, 23, 24, 25]. The details of these developments may be found in the articles referred to above. However, we give a brief description of the derivation of the fuzzy normal forms here.

2.3.1 New Construction of the Truth Tables

Truth Tables can be newly constructed for the derivation of Fuzzy Normal Forms as explained in Türkşen [21] and Türkşen *et al.* [27, 28] where three related but different explanations are given for this purpose.

For example, a newly constructed Truth Table is shown for the derivation of FDNF and FCNF of "A AND B" in Table 5. The new construction of the Truth Table for this purpose is based on the natural language expressions as shown below. Let us consider the two fuzzy sets A and B defined over the base axis X and a generic element $x \in X$, where $X = X_1 \cup X_2$. Let the A(x) = a and B(x) = b, $a, b \in [0, 1]$, where $a \geq b$ representing all cases of $x \in X_1$ and $a < b$ representing all cases $x \in X_2$.

Now the following linguistic expressions can be stated in order to cover all possible cases. For all $x \in X_1$:

(1.1.1) $x \in X_1$ is in the infinite-valued (fuzzy) set A with a membership degree $a \geq b$ is true, T, $t = 1$;
i.e., $x \in X_1$ isr A, $a \geq b$, isv T, $t = 1$.

(1.2.1) $x \in X_1$ is in the infinite valued (fuzzy) set A with a membership degree $a < b$ is false, F, $t = 0$;
i.e., $x \in X_1$ isr A, $a < b$, isv F, $t = 0$.

(2.1.1) $x \in X_1$ is in the infinite-valued (fuzzy) set B with a membership degree $a \geq b$ is true, T, $t = 1$;
i.e., $x \in X_1$ isr B, $a \geq b$, isv T, $t = 1$.

(2.2.1) $x \in X_1$ is in the infinite valued (fuzzy) set B with a membership degree $a < b$ is false, F, $t = 0$;
i.e., $x \in X_1$ isr B, $a < b$, isv F, $t = 0$.

For all $x \in X_2$, anologous expressions can be written, but let us just write the short hand notation only. That is, we have for all $x \in X_2$:

(1.1.2) x , X_2 isr A, $a < b$, isv T, $t = 1$.

(1.2.2) x , X_2 isr is A, a \$ b, isv F, $t = 0$.

(2.1.2) x , X_2 isr B, a < b, isv T, $t = 1$.

(2.2.2) x , X_2 isr B a \$ b, isv F, $t = 0$.

These two sets of linguistic expressions form the bases upon which one can newly construct the Truth Table for the definition of the combined fuzzy set "A AND B" in $S_{[0,1]}$ $L_{\{0,1\}}$ theory. The application of the Normal Form Derivation Algorithm [see 21] give us the derivation of FDNF and FCNF expressions for "A AND B" from the newly constructed Truth Table, Table 5, as follows:

$$\text{FDNF (A AND B)} = \begin{cases} (\text{A 1 B}),\ \text{A g B},\ \text{is } T, \\ (\text{B 1 A}),\ \text{A d B is } T; \end{cases}$$

$$\text{FCNF (A AND B)} = \begin{cases} (\text{A c B) 1 (c(A) c B) 1 (A c c(B))},\ \text{A g B is } T, \\ (\text{B c A) 1 (B c c(A) 1 (c(B) c A)},\ \text{A d B is } T. \end{cases}$$

It is to be noted that these are essentially duplicates of classical DNF and CNF expressions, respectively. The duplicate terms are realized due to the fact that one now has to take into account the two cases corresponding to A g B and A d B. It should be noted that in previous writings [21, 27, 28] a clear distinction between the two cases were not made explicit. This separation is needed because either x , X_1 or x , X_2 but not both, i.e., it is an "Exclusive OR" case.

In an analogous manner, one can derive the FDNF and FCNF expressions for the combined fuzzy set of "A OR B" to be:

$$\text{FDNF (A OR B)} = \begin{cases} (\text{A 1 B) c (c(A) 1 B) c (A 1 c(B))},\ \text{A g B is } T, \\ (\text{B 1 A) c (B 1 c(A)) c (c(B) 1 A)},\ \text{A d B is } T; \end{cases}$$

$$\text{FCNF (A OR B)} = \begin{cases} (\text{A c B}),\ \text{A g B is } T, \\ (\text{B c A}),\ \text{A d B is } T, \end{cases}$$

Again the duplicate terms are due to the cases of A g B and A d B. Hence for all the remaining fourteen expressions, i.e., FDNF and FCNF expression, can be obtained to be the duplicate of the classical DNF and CNF expressions shown in Table 2.

It should be further noted that derivation of these normal forms are based on *De Morgan Laws with the involutive standard negation and with the assumptions of monotonicity and boundary* conditions only. Hence, they are applicable to pseude *t*-norms and co-norms as well as *t*-norm and co-norms in general. Naturally, for the case of *t*-norms and *t*-conorms, we do not have to worry about the order of A g B and A d B. Thus we have only one set of formulas for all cases.

Therefore, it is straight forward to show that:

(i) these expressions reduce in *form only* to the classical expressions shown in Table 2 for all *t*-norms and *t*-conorms with FDNF (.) … FCNF (.). In particular for (Max, Min, St Com) De Morgan triple we get the particular characteristic relation that FDNF (.) f FCNF (.) [20].

(ii) these expressions further reduce to classical expressions when A and B are crisp, two-valued, sets with the particular characteristic relation that DNF (.) / CNF (.).

Therefore, the FDNF and FCNF expressions derived from newly constructed "Truth Table" for the $S_{[0,1]}$ $L_{\{0,1\}}$ theory are generalizations of the classical normal forms that take into account the two cases A d B or B f A that are realized in infinite-valued (fuzzy) set membership assignment while keeping the two-valued verity assignment of truthood.

2.4 $S_{\{0,1\}}$ $L_{[0,1]}$ $S_{[0,1]}$ $L_{[0,1]}$ Theories

In $S_{\{0,1\}}$ $L_{[0,1]}$ the set membership assignments are two-valued where as the verity assignments of truthood are in [0,1]. Next in $S_{[0,1]}$ $L_{[0,1]}$ the set membership assignments and the verity assignments of truthood are in [0, 1]. Currently, the normal forms of these two theories are being investigated and will be reported in a future paper.

3. Discussions

Each one of these theories help us to develop approximate reasoning schemas from more specific to more general. It is clear that the inference rules of two valued set and logic theory, i.e., $S_{\{0,1\}}$ $L_{\{0,1\}}$ theory, are well known. The inference rules of the Myopic $F(S_{\{0,1\}}$ $L_{\{0,1\}})$, theory are reasonably investigated in current literature, e.g., Dubois and Prade [6] Trillas and Valverde [19], etc.

But the inference rules of the $S_{[0,1]} L_{\{0,1\}}$ must be investigated in detail for use in approximate easoning schemas. Partial answers are provided in Tűrkşen [21, 22, 23, 24, 25] and Tűrkşen, *et.al*, [27, 28]. Based on fuzzy normal forms of $S_{[0,1]} L_{\{0,1\}}$ theory, three of the significant results are summarized here as follows:

(i) The laws of "Fuzzy Middle" and "Fuzzy Contradiction" can be stated and proved to hold as a matter of degree [28].

(ii) The laws of "Conservation" holds absolutely in a new form, [28] i.e.,

$$\mu[\text{FCNF (A OR C(A)}] + \mu[\text{FDNF (A AND C (A)}] = 1, \text{ and}$$
$$\mu[\text{FDNF (A OR C(A)}] + \mu[\text{FCNF (A AND C(A)}] = 1.$$

(iii) The expressions of "Re-affirmation" and "Re-negation" hold as a matter of degree in sensor fusion [28].

It is expected that further research results of $S_{[0,1]} L_{\{0,1\}}$ theory will be reported for a well founded development of the theory with due regard to FDNF and FCNF.

Finally much work and investigation is called for $S_{\{0,1\}} L_{[0,1]}$ and $S_{[0,1]} L_{[0,1]}$ theories which are yet to be explored extensively.

Table 1. Meta-Linguistic Expressions of Combined Concepts for any A and B

Number	Meta-Linguistic Expressions
1	UNIVERSE
2	EMPTY SET
3	A OR B
4	NOT A AND NOT B
5	NOT A OR NOT B
6	A AND B
7	A IMPLIES B
8	A AND NOT B
9	A OR NOT B
10	NOT A AND B
11	A IF AND ONLY IF B
12	A EXCLUSIVE OR B
13	A
14	NOT A
15	B
16	NOT B

Table 2. Classical Disjunctive and Conjunctive Normal Forms, where ∩ is a conjunction, ∪ is a disjunctive and c is a complementary operator

No.	Disjunctive Normal Forms	Conjunctive Normal Forms
1	(A1B) c (A1c(b)) c (c(A)1B) c (c(A)1c(B))	I
2	i	(AcB) 1 (Acc(B)) 1 (c(A)cB) 1 (c(A)cc(B))
3	(A1B) c (A1c(B)) c (c(A)1B)	(AcB)
4	(c(A)1c(B))	(Acc(B)) 1 (c(A)cB) 1 (c(A)cc(B))
5	(A1c(B)) c (c(A)1B) c (c(A)1c(B))	(c(A)cc(B))
6	(A1B)	(AcB) 1 (Acc(B)) 1 (c(A)cB)
7	(A1B) c (c(A)1B) c (c(A)1c(B))	(c(A)cB)
8	(A1C(B))	(AcB) 1 (Acc(B)) 1 (c(A)cc(B))
9	(A1B) c (A1c(B)) c (c(A)1c(B))	(Acc(B))
10	(c(A)1B)	(AcB) 1 (c(A)cB) 1 (c(A)cc(B))
11	(A1B) c (c(A)1c(B))	(Acc(B)) 1 (c(A)cB)
12	(A1c(B)) c (c(A)1B)	(AcB) 1 (c(A)cc(B))
13	(A1B) c (A1c(B))	(AcB) 1 (Acc(B))
14	(c(A)1B) c (c(A)1c(B))	(c(A)cB) 1 (c(A)cc(B))
15	(A1B) c (c(A)1B)	(AcB) 1 (c(A)cB)
16	(A1c(B)) (c(A)1c(B))	(Acc(B)) 1 (c(A)cc(B))

Table 3. Classical truth table interpretations of "A AND B"

Truth assignments to meta-linguistic variables A	B	Truth assignments to the meta-linguistic expression "A AND B"	Primary classical conjunctions
T(A)	T(B)	T(A AND B)	**A ∩ B**
T(A)	F(B)	F(A AND B)	**A ∩ c(B)**
F(A)	T(B)	F(A AND B)	**C(A) ∩ B**
F(A)	F(B)	F(A AND B)	**c(A) ∩ c(B)**

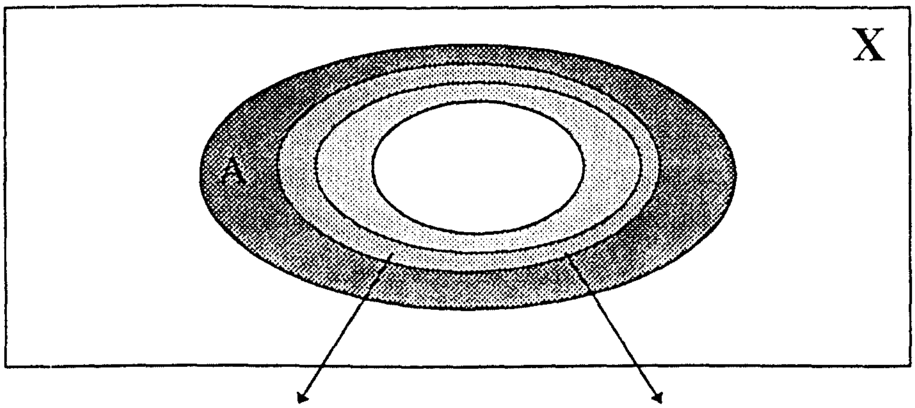
Table 4. Axioms of Classical & Set Operations

Involution	c(c(A)) = A
Commutativity	AcB = BcA
	A1B = B1A
Associativity	(AcB)cC = Ac(BcC)
	(A1B)1C = A1(B1C)
Distributivity	(AcB)cC = Ac(BcC)
	(A1B)1C = A1(B1C)
Idempotence	AcA = A
	A1A = A
Absorption	Ac(A1B) = A
	A1(A cB) = A
Absorption by X and i	AcX = X
	A1i = i
Identity	Aci = Professor A.
	Mandelis A1X = A
Law of contradiction	A1c(A) = i
Law of excluded middle	Acc(A) = X
De Morgan's laws	c(A1B) = c(A)cc(B)
	c(AcB) = c(A)1c(B)

A			B			A AND B
(1.1.1)	a ≥ b	T	(2.1.1)	a ≥ b	T	T
(1.1.1)	a ≥ b	T	(2.2.1)	a < b	F	F
(1.2.1)	a < b	F	(2.1.1)	a ≥ b	T	F
(1.2.1)	a < b	F	(2.2.1)	a < b	F	F
(1.1.2)	a < b	T	(2.1.2)	a < b	T	T
(1.1.2)	a < b	T	(2.2.2)	a ≥ b	F	F
(1.2.2)	a ≥ b	F	(2.1.2)	a < b	T	F
(1.2.2)	a ≥ b	F	(2.2.2)	a ≥ b	F	F

Table 5. The newly constructed "Truth Table" of "A AND N(A)" where a, b ∈ [0, 1].

SET AND LOGIC



SET DESCRIPTION, D

s ::= $x \in X$ isr A, $\mu_A(x) \in [0,1]$, $s \in S_D$

Assignment of $x \in X$ to a fuzzy information granule $A \in \mathcal{A}$, any linguistic variable, within the context of fuzzy proposition "X isr A" in set description statements, $s \in S_D$.

TRUTHOOD VERIFICATION, V

t ::= $s \in S_D$ isv T, $\mu_T(s) \in [0,1]$, $t \in S_V$

Assignment of $s \in S_D$, to a fuzzy information granule $T \in \mathcal{T}$, the particular linguistic variable i.e., truth qualification variable, within the context of fuzzy proposition "S_D isv T in truthhood statements, $t \in S_V$.

KNOWLEDGE REPRESENTATION(K)

K=(D,V)

Figure 1. Fuzzy Venn Diagram

REFERENCES

[1] Bergmann, M., Moor, J., Nelson, J.A., (1988) The Logic Book, McGraw Hill, New York.

[2] Bezdek, J.C. (1973) "Fuzzy Mathematics in Pattern Recognition", Ph.D., Thesis, Cornell University.

[3] Bezdek, J.C. (1981) "Pattern Recognition with Fuzzy Objective Function Algorithms", Plenum Press, New York.

[4] Bilgic, T. and Türkşen, I.B. (1997) "Elicitation of Membership Functions: How Far Can Theory Take Us", Proceedings of IEEE-SMC Conference, July 1997, Barcelona, Spain, 1321-1325.

[5] Copi, I.M., *Introduction to Logic* Macmillan Pub. Co., Inc., 1982.

[6] Dubois, D., Prade, H., (1984), "Fuzzy Logics and the Generalized Modus Ponens Revisited", Inter. Journal of Cybernetics and Systems, 283-331.

[7] Emami, M.R., Türkşen, I.B. and Goldenberg, A.A. (1981) "A Unified Parameterized Formation on Reasoning Process in Fuzzy Modeling and Control", Fuzzy Sets and Systems, (to appear).

[8] Emami, M.R., Türkşen, I.B. and Goldenberg, A.A. (1997) "Fuzzy Clustering and Fuzzy Modeling", IEEE-Trans on Fuzzy Systems, (to appear).

[9] Fukuyama, Y., Sugeno, M. (1989) "A New Method of Choosing the Number of Clusters for Fuzzy C-Means", Proceedings of the 5th Fuzzy Systems Symposium (in Japanese), 247-250.

[10] Katz, M. (1982) "Real-Valued Models with Metric Equality in Uniformly Continuous Predicates", Journal of Symbolic Logic, **47**, 772-792.

[11] Ludwig, G. (1981) "Impressions in Physics", in A. Hartkamper and H.J. Schmidt (eds), *Structure and Approximation in Physical Theories*, Plenum Press, New York.

[12] Menger, K. (1951), Probabilistic Theory of Relations", Proc. of Nat. Acad. Sci., USA, **37**, 178-180.

[13] Nakanishi, H., Türkşen, I.B., and Sugeno, M., (1993) "A Review and Comparison of Six Reasoning Methods", Fuzzy Set and Systems, **57**, 257-294.

[14] Pultr, A. (1984) "Fuzziness and Fuzzy Equality", in H.J. Skala, S. Termini and E. Trillas (eds), *Aspects of Vagueness*, D. Reidel Publ. Co., 119-136.

[15] Ruspini, E. (1982) "Recent Developments in Fuzzy Clustering" in R.R. Yager (ed) *Fuzzy Sets and Possibility Theory: Recent Developments*, Pergamon Press, 133-147.

[16] Ruspini, E. (1969) "A New Approach to Clustering", Information and Control, **15**, 22-32.

[17] Russel, B. (1923) "Vagueness", Australian Journal of Philosophy, **1**, 83-92.

[18] Sugeno, M. and Yasukawa, T. (1993) "A Fuzzy-Logic Based Approach to qualitative Modeling, IEEE-Trans. Fuzzy Systems, **1**, 1, 7-31.

[19] Trillas, E., and Valverde, E., (1985), "On Mode and Implication in Approximate Reasoning", in:W. Bandler, M.M. Gupta, A. Kandel, and J.B. Kiska (eds.), Approximate Reasoning in Expert Systems, Elsevier Sci. B.V. (North Holland) 157-167.

[20] Türkşen, I.B., (1986), "Interval-Valued Fuzzy Sets based on Normal Forms", FSS, **20**, 191-210.

[21] Türkşen, I.B. (1994) "Fuzzy Normal Forms", FSS, 253-266.

[22] Türkşen, I.B. (1994) "Interval-Valued Fuzzy Sets and 'Compensatory AND'", FSS, **51**, 295-307.

[23] Türkşen, I.B. (1995) "Type I and 'Interval Valued' Type II Fuzzy Sets and Logics", in P.P. Wang (ed), *Fuzzy Theory and Technology*, **III**, 31-82.

[24] Türkşen, I.B. (1996) "Non-Specificity and Internal Valued Fuzzy Sets", FSS, 87-100

[25] Türkşen, I.B. (1998) "Type I and Type II Fuzzy System Modeling", FSS, (to appear).

[26] Türkşen, I.B. and Zarandi, M.H.F. (1998) "Fuzzy System Models for Aggregate Scheduling Analysis", IJAR, (to appear).

[27] Türkşen, I.B. and Kandel, A. And Y-Q. Zhang (1998). "Universal Truth Tables and Normal Forms", IEEE-Fuzzy Systems, **6**(2), 295-303.

[28] Türkşen, I.B. and Kandel, A. And Y-Q. Zhang (1999) . "Normal Forms of Fuzzy Middle and Fuzzy Contradiction", IEEE-SMC (to appear).

[29] Zadeh, L.A. (1971) "Similarity Relations and Fuzzy Orderings", Information Sciences, **3**, 177-200.

[30] Zadeh, L.A. (1973) "Outline of a New Approach to the Analysis of Complex Systems and Decisions Processes", IEEE-Trans. Syst. Man. Cybern, **3**, 28-44.

[31] Zadeh, L.A. (1997) "Toward a Theory of Fuzzy Information Granulation and Its Centrality in Human Reasoning and Fuzzy Logic", Fuzzy Sets and Systems, **90**, 111-127.