OPTIMAL TEST FUNCTION PETROV-GALERKIN METHOD

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INTRODUCTION

Transport in the environment is described by the convection diffusion equation or the transport equation. The transport equation is a mixed type of partial differential equation which has both hyperbolic and parabolic characteristics. A difficulty in the analysis of the transport equation lies in that the solution strategy should be chosen according to the characteristic of the given problem. That is, if the parabolic feature of the problem is stronger than the hyperbolic feature, then a numerical modeler does not have to concern numerical oscillations because the dispersion term naturally smoothes the solution. However, in the reversed case, the modeler needs to introduce a special dissipative tool for the convection term because of a possible generation of the steep gradient in the numerical solution. In the latter case, mostly, the upwinding concept should be introduced to protect the solution from the downwind contamination (Christie et al., 1976 ; Heinrich et al., 1977 ; Brooks and Hughes, 1982 ; Westerink and Shea, 1989). This can be easily achieved, if the finite difference method is employed, by using upwind type schemes such as the backward difference scheme, MacCormack scheme, or Beam and Warming scheme.

The conventional finite element method (or Bubnov Galerkin method) is very powerful for the boundary value problem in which governing equations are self-adjoint elliptic partial differential equations. However, the Galerkin method is not so efficient for problems which have non-self-adjoint equations such as convection-diffusion problems. This equation can be characterized by a non-dissipative convection process and a dissipative diffusion process. When dissipative processes dominate the transport phenomenon, one can get a good solution using the conventional Galerkin method. However, when the non-dissipative convection process is dominant, the partial differential equation behaves as a first-order hyperbolic partial differential equation which describes a pure wave propagation with a finite celerity. Therefore, the numerical solution may form or maintain a sharp front. When the conventional Galerkin method with linear basis functions is applied to this problem, oscillations or dissipations are encountered in the numerical solution. Specifically, in the one-dimensional case, the finite element scheme can be easily shown to become centered difference finite difference scheme. Since the numerical errors arise from the symmetric treatment of the convection terms in the conventional finite element method, the test functions are modified in order to give more weight in the upwind direction as commonly done in the finite difference method. This technique is referred to as "the Petrov-Galerkin (PG) method."

In this paper, a new Petrov-Galerkin method, termed as "Optimal Test Function Petrov-Galerkin (OTFPG) Method" is proposed. The test function of this numerical

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method changes its shape depending upon relative strength of the convection to the diffusion. A numerical experiment is carried out to demonstrate the performance of the proposed method.

FINITE ELEMENT FORMULATION

Consider the following one-dimensional transport equation:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} - u\frac{\partial c}{\partial x} \tag{1}$$

which is a parabolic partial differential equation. In general, one encounters the numerical instabilities due to convection term when the mesh size (Δx) is such that the grid Peclet number

$$Pe \equiv \frac{u\Delta x}{D} > 2 \sim 5 \tag{2}$$

which indicates the relative dominance of the convection to the diffusion. The weighted residual from of eq.(1) is written as

$$\int \left\{ \left(\frac{\partial c}{\partial t}\right) w + D\frac{\partial c}{\partial x}\frac{\partial w}{\partial x} \right\} dx = D\frac{\partial c}{\partial x} \Big|_{boundary} \tag{3}$$

where w is a test function. After the assembly process, one can get a global system of matrix differential equation such as

$$M\frac{dc}{dt} - (A^v - A^d)\ c = P \tag{4}$$

where c = vector of nodal unknowns, M = mass matrix, A^v = convection stiffness matrix, A^d = diffusion stiffness matrix, and P = convective and diffusive boundary flux forcing vector. In the conventional Galerkin method, the basis function of the trial solution is the same as the basis function of the test function. However, the PG method employs the test function whose basis function is different from the basis function of the trial solution. The numerical oscillations encountered in the numerical analysis of the convection dominant flow comes from the non-symmetry of the stiffness matrix due to A^v. Therefore, the PG formulation was devised to recover the symmetry of the stiffness matrix to some extent.

OPTIMAL TEST FUNCTION METHOD

Before introducing the OTFPG method, it may be worth to Optimal Test Function (OTF) method by Celia et al. (1989a). Consider an operator form of the one-dimensional transport equation such as

$$Lc \equiv D\frac{\partial^2 c}{\partial x^2} - u\frac{\partial c}{\partial x} = \frac{\partial c}{\partial t} - f(x), \qquad 0 \le x \le l \tag{5}$$

with the following boundary conditions:

$$c(0) = g_o \quad \text{and} \quad c(l) = g_l \tag{6}$$

The weak form of eq.(15) is

$$\int_0^l (Lc)\, w(x) dx = \int_0^l \left(\frac{\partial c}{\partial t} - f(x,t) \right) w(x)\, dx \tag{7}$$

where $w(x)$ is a test function. If we discretize the domain into E sub-intervals with

E+1 nodal points, then eq.(7) can be written as

$$\int_0^l (Lc)\,w(x)dx \;=\; \sum_{j=0}^{E-1}\int_{x_j}^{x_{j+1}}(Lc)\,w(x)\,dx \tag{8}$$

Integrating by parts the RHS of eq.(7) after applying eq.(15), we have

$$\int_0^l (Lc)\,w(x)dx \;=\; \sum_{j=0}^{E-1}\left\{\left[Dw\frac{dc}{dx}-uwc\right]_{x_j}^{x_{j+1}} - \int_{x_j}^{x_{j+1}}\left(D\frac{dc}{dx}-uc\right)\frac{dw}{dx}\,dx\right\} \tag{9}$$

Another integration by parts of the second term of the RHS of eq.(9) yields

$$\int_0^l (Lc)\,w(x)dx=\sum_{j=0}^{E-1}\left\{\left[Dw\frac{dc}{dx}-D\frac{dw}{dx}\,c-uwc\right]_{x_j}^{x_{j+1}}+\int_{x_j}^{x_{j+1}}(L^{*}w)c(x)dx\right\} \tag{10}$$

where L^{*} is the formal adjoint of L. If one takes a test function which satisfies $L^{*}w=0$ within each element, the following relationship should be hold:

$$\int_0^l (Lc)\,w(x)dx \;=\; \sum_{j=0}^{E-1}\left\{\left[Dw\frac{dc}{dx}-Dw\frac{dc}{dx}-uwc\right]_{x_j}^{x_{j+1}}\right. \tag{11}$$

Then, we have the following final form of equation from the OTF method:

$$\int_0^l (Lc)\,w(x)dx \;=\; \sum_{j=0}^{E-1}\left\{\left[Dw\frac{dc}{dx}+uw\right]_{x_j}c_j - [Dw]_{x_j}\frac{dc_j}{dx}\right\} + \left[(-D\frac{dw}{dx}-uw)c+(Dw)\frac{dc}{dx}\right]_0^l \tag{12}$$

where [ ] is a jump operator defined as

$$[\,\cdot\,]_{x_j} \;=\; [\,\cdot\,]_{x_j^-} \;-\; [\,\cdot\,]_{x_j^+} \tag{13}$$

If explicit forms of $w(x)$ can be derived, eq.(12) leads to $(2E+2)\times(2E+2)$ algebraic equations, a 5 bandwidth matrix. The unknowns are essentially nodal values of function c_j and their derivatives, i.e., $[c_j, dc_j/dx]_{j=1}^{E}$. Notice that no approximation has been introduced in the formulation so far. However, the homogeneous adjoint equation exhibits non-constant coefficients and cannot be solved exactly, in general.

OPTIMAL TEST FUNCTION PETROV-GALERKIN METHOD

The main idea of the OTFPG method comes from the application the test function of the OTF method to the PG method. That is, instead of solving eq.(12), one can use the test function satisfying $L^{*}\omega=0$ from eq.(10) in the PG formulation. In order to obtain the test function, the following equation should be solved:

$$L^{*}w \equiv D\frac{d^2w}{dx^2}+u\frac{dw}{dx}=0 \tag{14}$$

which is a second-order homogeneous linear ordinary differential equation. We, thus, have two fundamental solutions which are linearly independent, i.e.,

$$w_1(x) = 1 \tag{15a}$$

$$w_2(x) = \exp[-(u/D)x] \tag{15b}$$

Any linear combination of these solutions can be a solution of eq.(14). However, we have the following two restrictions for w_i to be test functions:

$$w_1(x) = 1, \qquad w_1(\Delta x) = 0 \tag{16a}$$

$$w_2(x) = 0, \qquad w_2(\Delta x) = 1 \tag{16b}$$

After applying the fundamental solutions of eq.(14) into eq.(15), we have two particular solutions such as

$$w_1(x) = \frac{\exp(-ux/D) - \exp(-u\Delta x/D)}{1-\exp(-u\Delta x/D)} \quad 0 \le x \le \Delta x \tag{17a}$$

$$w_2(x) = \frac{1-\exp(-ux/D)}{1-\exp(-u\Delta x/D)} \quad 0 \le x \le \Delta x \tag{17b}$$

Fig.1 shows test functions from eq.(17), which have various shapes depending upon the relative strength of convection term to the dispersion term. When u/D = 0.1 m^{-1}, the test functions in Fig.1(a) are nearly the same as linear shape functions, and when u/D is increased up to 2000.0 m^{-1}, the test functions in Fig.9(d) have a shape like a step function.

APPLICATION EXAMPLE

A convection-dominated transport problem (D = 2.5 × 10^{-5} m^2/day, u = 0.05 m/day) is solved to demonstrate the applicability of the proposed OTFPG method. An initial condition of $c(x, t=0) = 0$ is used, and such boundary conditions as

$$\left(uc - D\frac{\partial c}{\partial x} \right)_{x=0} = uc_{feed} \tag{18a}$$

$$c(x \to \infty, t) = 0 \tag{18b}$$

are used at the upstream and the downstream boundaries, respectively. Values of $c_{feed} = 1.0$, $\Delta t = 1.0$ day, and $\Delta x = 0.05$ m are used in the computation, resulting Cr = 1 and Pe = 100. The grid Peclet number is large enough to cause oscillations in the numerical solutions. Computed results are presented in Fig.2, which shows excellent agreement between the analytical and numerical solutions. This illustrates the capability of the numerical method which removes fictitious oscillations around the high-gradient area in the solution while maintaining a quite steep front.

CONCLUSIONS

A new PG method, termed as optimal test function Petrov-Galerkin method, is introduced in this paper. The test functions of the new method are functions of a grid Peclet number so that they change their shapes depending upon the characteristic of the transport problem. The new method was applied to a convection-dominated transport problem, and was found to yield excellent agreement between the analytical and the numerical solutions. The present study explicitly indicates that the test functions of the PG method should be a function of a grid Peclet number once the Courant restriction is satisfied in the convection-dominated transport problem.

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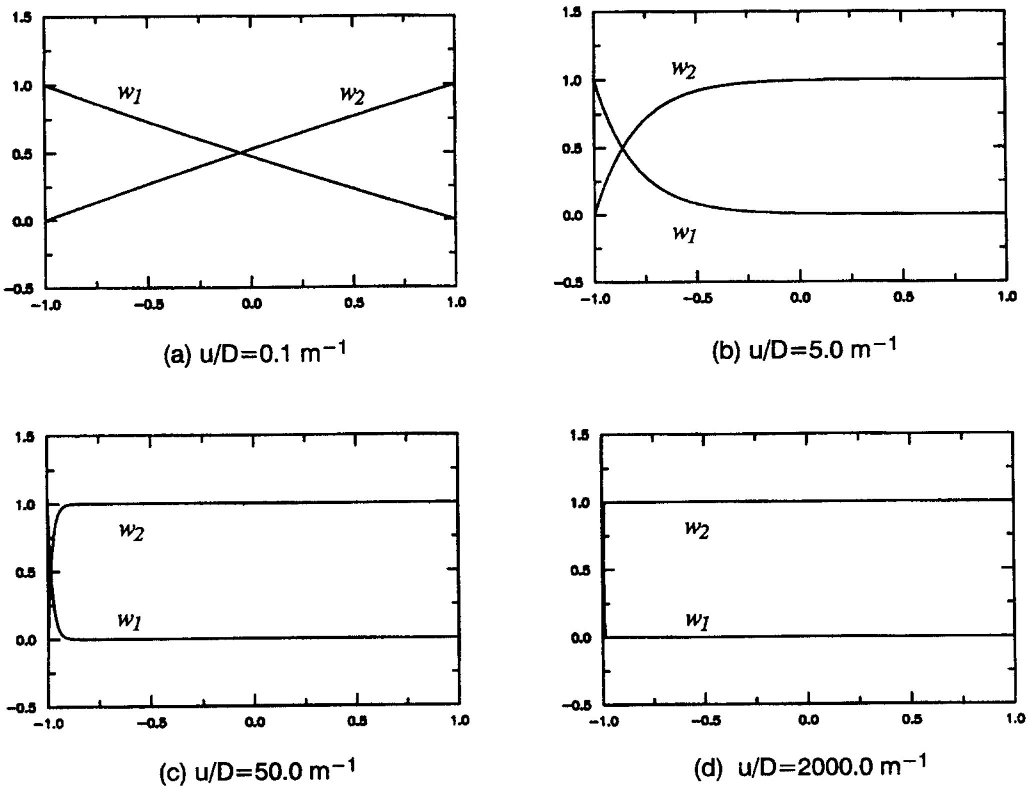

Figure 1. Test Functions of Optimal Test Function Petrov−Galerkin Method

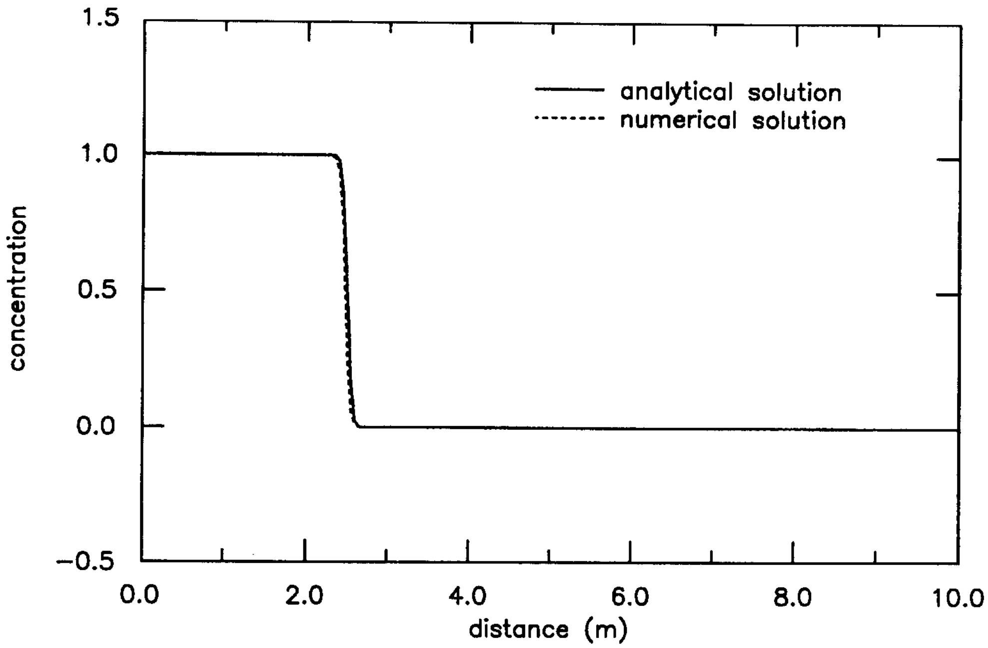

Figure 2 (a). Concentration Profile at 50 days

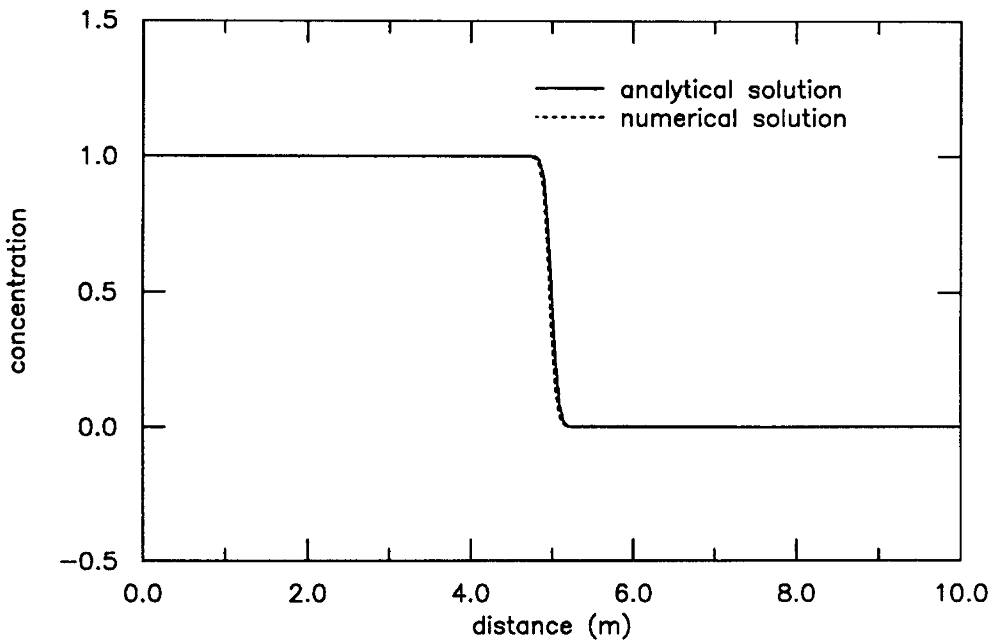

Figure 2 (b). Concentration Profile at 100 days