SMOOTH NONPARAMETRIC ESTIMATION OF MEAN RESIDUAL LIFE[1)]

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ABSTRACT

In this paper we propose smooth nonparametric estimator of Mean Residual Life(MRL) based on a complete sample. This estimator is constructed using estimator of cumulative failure rate which is derived as the maximum likelihood estimate of cumulative failure rate in the class of distributions which have piecewise linear failure rate functions between each pair of observations. We derive the asymptotic properties of the our estimator. The proposed estimator is compared with previously known estimator by Monte Carlo study.

1. Introduction

Let F be a continuous life distribution(i.e., $F(x)=0$ for $x \le 0$) with the finite first moment and let X be a nonnegative random variable with distribution F. The mean residual life(MRL) function $e(x)$ is defined as

$$e(x) = E(X-x \mid X > x), \tag{1.1}$$

The MRL is the expected remaining lifetime, $X-x$, given that the item has survived to time x. The MRL function $e(x)$ in (1.1) can also be written as

$$e(x) = \frac{\int_x^\infty \overline{F}(u)\,du}{\overline{F}(x)},$$

where $\overline{F}(x) = 1-F(x)$ is the reliability function.

The MRL function plays a very important role in the area of engineering, medical science, survival studies, social sciences, and many other fields. The MRL is used by engineers in burn-in studies, setting maintenance policies, and in comparison of life distributions of different systems. Social scientists use MRL,

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1) This work was partially supported by the Basic Science Research Institute Program, Ministry of Education, 1997, Project No. BSRI-97-1415.

also called as inertia, in studies of lengths of wars, duration of strikes, job mobility etc. Medical researchers use MRL in lifetime experiments under various conditions. Actuaries apply MRL to setting rates and benefits for life insurance.

Hall and Wellner(1981) derive that all MRL functions associated with distributions having a finite mean must satisfy three conditions:

$$e(x) \geq 0, \quad e'(x) \geq -1, \quad \int_0^{\infty} \frac{1}{e(x)}\, dx = \infty.$$

See also Bhattacharjee(1982) for another characterization of MRL. Knowledge of the MRL function completely determines the reliability function, via the relation

$$\overline{F}(x) = \frac{e(0)\exp\left\{-\int_0^x [e(u)]^{-1} du\right\}}{e(x)}, \quad x \geq 0. \tag{1.2}$$

Kotz and Shanbhag(1980) derive a generalized inversion formula for distributions that are not necessarily life distributions. Hall and Wellner(1981) have an excellent discussion of (1.2).

We consider nonparametric estimation of mean residual life $e(x)$. The estimation of $e(x)$ is very useful in practice and a few nonparametric estimation procedure have been suggested in the literature. The empirical MRL estimator, $e_n(x)$, proposed by Yang (1978) can be obtained by replacing $\overline{F}$ of equation (1,1) with the empirical distribution as shown by the following equation

$$e_n(x) = \begin{cases} \frac{1}{n-k} \sum_{i=k+1}^{n} (X_{(i)} - x) & \text{for } X_{(k)} \leq x \leq X_{(k+1)} \\ 0 & \text{for } x \geq X_{(n)} \end{cases}$$

for $k = 0, 1, \cdots, n-1$, $X_{(0)} = 0$ and $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ is the order statistics of a random sample $X_1, X_2, \cdots, X_n$. For case of ties, the estimator is lightly modified as given in Guess and Proschan(1988). Yang(1978) has shown that $e_n(x)$ is no unbiased estimator of $e(x)$. However, Yang(1978) proved that $e_n(x)$ is asymptotically unbiased, uniformly strong consistent, and converges in distribution to a Gaussian process. Mi(1994) propose an estimator which has a mean residual life function satisfying the same requirement on the shape for any member of classes which have decreasing, increasing or upside-down shaped mean residual life.

In this paper we propose nonparametric estimator of $e(x)$ based on a complete sample. This estimator is constructed using the estimator $\hat{\Lambda}(x)$, where $\hat{\Lambda}(x)$ is derived as the maximum likelihood estimate of cumulative failure rate $\Lambda(x)$ in the class of distributions which have piecewise linear failure rate functions between each pair of observations. The resulting estimators of $e(x)$ is smooth. We derive the asymptotic properties of the our estimators. Monte Carlo simulation are conducted to investigate the performance of our new nonparametric estimators of Mean Residual Life. Smooth nonparametric estimation procedures are discussed in Section 2. Section 3 is devoted to proof of main theorems. Results of Monte carlo simulation are presented in Section 4.

2. The Proposed Estimator

Let K denote a positive integer and let $\xi_1, \cdots, \xi_K$ be a (simple) knot sequence in $[0, \infty)$ where $0 < \xi_1 < \cdots < \xi_K < \infty$. Let S denote the collection of piecewise continuous linear functions s on $[0, \xi_K]$ such that the restriction of s to each of the intervals $[0, \xi_1], [\xi_1, \xi_2], \cdots, [\xi_{K-1}, \xi_K]$ is a linear function. Then S is the $(K+1)$-dimensional vector space and has a basis $B_0, B_1, \cdots, B_K$. (See de Boor, 1978)

Let Θ denote the collection of all column-vector $\theta = (\theta_0, \theta_1, \cdots, \theta_K)^t \in R^{K+1}$ such that $\sum_{j=0}^{K} \theta_j B_j(x) > 0$. Given $\theta \in \Theta$, we approximate the failure rate function by

$$\lambda(x : \theta) = \sum_{j=0}^{K} \theta_j B_j(x) \tag{2.1}$$

over the interval $0 \le x \le \xi_K$. For the approximation (2.1), the corresponding cumulative failure rate and reliability function are given by

$$\begin{aligned} \Lambda(x : \theta) &= \sum_{j=0}^{K} \theta_j \int_0^x B_j(u) du, \\ \overline{F}(x : \theta) &= \exp\left(- \sum_{j=0}^{K} \theta_j \int_0^x B_j(u) du \right). \end{aligned}$$

We determine the coefficients of the linear combination by maximizing the likelihood function. Let $X_1, X_2, \cdots, X_n$ be a random sample from a life distribution F with a density function f. Then the log-likelihood function corresponding to the approximation (2.1) is determined by

$$l(\theta) = \sum_{i=1}^{n} \ln \lambda(x_i : \theta) - \sum_{i=1}^{n} \int_0^{x_i} \lambda(u : \theta) du.$$

We place the knots $\xi_1 \le \xi_2 \le \cdots \le \xi_K$ by distinct failure time $X'_{(1)} < X'_{(2)} < \cdots < X'_{(K)}$ which are different sorted values of $X_1, X_2, \cdots, X_n$. Using $B_j(\xi_j) = 1$, and 0 at other knots gives the minimizing solution $\hat{\theta}_0=0$ and

$$\hat{\theta}_j=\frac{m_j}{\sum_{i=1}^{n}\int_0^{x_i}B_j(u)du}, \qquad \text{for} \quad j=1,2,\cdots,K,$$

where m_j is the number of failure times equal $X'_{(j)}$. Then the estimate of the cumulative failure rate is

$$\hat{\Lambda}(x)=\sum_{j=1}^{K}\frac{m_j\int_0^{x}B_j(u)du}{\sum_{i=1}^{n}\int_0^{x_i}B_j(u)du}. \tag{2.2}$$

The estimator (2.2) is a non-negative differentiable monotone increasing function of x on the interval $[0, \xi_K]$ and thus the estimate of the reliability,

$$\hat{\overline{F}}(x)=\exp\left(-\sum_{j=1}^{K}\frac{m_j\int_0^{x}B_j(u)du}{\sum_{i=1}^{n}\int_0^{x_i}B_j(u)du}\right)$$

is a differentiable monotone decreasing function on this interval. With this estimator of $\overline{F}$, we define the estimator $\hat{e}(x)$ of $e(x)$ as

$$\hat{e}(x)=\frac{\overline{X}-\int_0^{x}\hat{\overline{F}}(u)du}{\hat{\overline{F}}(x)} \tag{2.3}$$

where $\overline{X}$ is the sample mean. By the definition, this is a continuous estimator of $e(x)$. The asymptotic properties of this estimator are obtained under the following assumptions.

A1. F is continuous

A2. $T_F\leq\infty$, where $T_F=\sup\{x: \overline{F}(x)>0\}$.

A3. $\mu_F=e(0)<\infty$.

Now we have the following results. We present the results in this section and provide proofs in Section 3.

THEOREM 2.1 Let $x\in[0, T]$, $0<T<T_F$ and let assumptions A1-A3 be satisfied. Then the estimator $\hat{e}(x)$ is consistent for $e(x)$.

THEOREM 2.2 Let $T<\infty$ satisfying $\overline{F}(T)>0$ and let assumptions A1-A3 hold. Then the process $\{\sqrt{n}(\hat{e}(x)-e(x)) : x\in[0, T]\}$ converges weakly to a mean zero Gaussian process with covariance structure

$$\Gamma(x,y)=\frac{\overline{F}(x)\overline{F}(y)\sigma^2(x,1)-F(y)\overline{F}(x)\theta^2(x,1)}{(\overline{F}(x)\overline{F}(y))^2}$$

where $\theta(s,t)=E\{XI(s<F(x)\le t)\}$ and $\sigma^2(s,t)=Var\{XI(s<F(x)\le t)\}$. Here $I(\cdot)$ is the indicator function.

3. Proof of Theorems

Let $\Lambda_n(x)$ denote the empirical cumulative failure rate function defined by

$$\Lambda_n(x)=\sum_{j=1}^{K}\frac{I[\xi_j\le x]}{n(1-F_n(\xi_j-))}.$$

Then we can obtain the following Lemma 3.1 applying the technique in proof of theorem 1 of Klotz(1982).

LEMMA 3.1 Let $x\in[0,T]$, $0<T<T_F$ and let assumptions A1-A2 be satisfied. Then

$$\sup_{\{0\le x\le T\}}\sqrt{n}(\hat{\Lambda}(x)-\Lambda_n(x))\xrightarrow{p}0 \quad \text{as} \quad n\to\infty.$$

To obtain the asymptotic properties of our estimator, the proofs utilize the following Lemma 3.2.

LEMMA 3.2 Let $x\in[0,T]$, $0<T<T_F$ and let assumptions A1-A3 be satisfied. Then

$$\sup_{\{0\le x\le T\}}\sqrt{n}(\hat{\overline{F}}(x)-\overline{F}_n(x))\xrightarrow{p}0 \quad \text{as} \quad n\to\infty,$$

where $\overline{F}_n(x)$ is the empirical reliability function.

PROOF Write

$$\begin{aligned}\sqrt{n}(\hat{\overline{F}}(x)-\overline{F}_n(x)) &= \sqrt{n}\big(\exp(-\hat{\Lambda}(x))-\exp(-\Lambda_n(x))\big)\\ &\quad+\sqrt{n}(\exp(-\Lambda_n(x))-\overline{F}_n(x))\\ &= -\sqrt{n}(\hat{\Lambda}(x)-\Lambda_n(x))\exp(-\Lambda_n^*(x))\\ &\quad+\sqrt{n}(\exp(-\Lambda_n(x))-\overline{F}_n(x))\end{aligned}$$

where $\Lambda_n^*(x)=a_n(x)\hat{\Lambda}(x)+(1-a_n(x))\Lambda_n(x)$ with $0\le a_n(x)\le 1$, for $x\ge 0$. Note that a_n exists by the mean value theorem. The required result follows from Lemma 3.1 and the similar technique in proof of theorem 5 of Breslow and Crowley (1974). □

PROOF OF THEOREM 2.1 We can write

$$
\begin{aligned}
|\hat{e}(x)-e(x)| \le & \left|\frac{\overline{X}-\mu}{\widehat{\overline{F}}(x)}\right| + \left|\frac{\mu(\overline{F}(x)-\widehat{\overline{F}}(x))}{\widehat{\overline{F}}(x)\overline{F}(x)}\right| \\
& + \left|\frac{\int_0^t(\widehat{\overline{F}}(u)-\overline{F}(u))du}{\widehat{\overline{F}}(x)}\right| + \left|\frac{\int_0^t\overline{F}(u)du(\overline{F}(x)-\widehat{\overline{F}}(x))}{\overline{F}(x)}\right|.
\end{aligned}
\tag{3.1}
$$

Thus the right side of the above inequality (3.1) converges in probability 0 as $n\to\infty$ by Lemma 3.2 and Theorem 5.1 of Billingsley(1968). □

PROOF OF THEOREM 2.2 Using (1.1) and (2.3), write

$$
\begin{aligned}
\sqrt{n}(\hat{e}(x)-e(x)) &= \sqrt{n}(\hat{e}(x)-e_n(x))+\sqrt{n}(e_n(x)-e(x)) \\
&= \sqrt{n}(e_n(x)-e(x))+R_{1n}(x)+R_{2n}(x)
\end{aligned}
$$

where

$$R_{1n}(x)=-\sqrt{n}\left(\widehat{\overline{F}}(x)\ \overline{F}_n(x)\right)^{-1}\left(\widehat{\overline{F}}(x)-\overline{F}_n(x)\right)\int_x^\infty \overline{F}_n(u)du$$

and

$$R_{2n}(x)=-\sqrt{n}\left(\widehat{\overline{F}}(x)\ \overline{F}_n(x)\right)^{-1}\overline{F}_n(x)\int_0^x\left(\widehat{\overline{F}}(u)-\overline{F}_n(u)\right)du.$$

It follows from Lemma 3.2 and the classical weak convergence of the empirical process that $\sup R_{1n}(x)$ and $\sup R_{2n}(x)$ converge in probability 0 as $n\to\infty$. Thus the required result follows Theorem 1 of Yang(1978) and Theorem 4.1 of Billingsley(1968). □

4. Simulation Study

In this section, we perform a Monte Carlo simulation to investigate the performance of our new nonparametric estimators of Mean Residual Life. Simulations are performed on a super computer SP2 at Seoul National University using the programing language FORTRAN.

To investigate the performance of our new estimators, we generate random numbers from weibull and gamma distribution using the IMSL subroutines. From each of a number of specified distributions, chosen so as to have a variaty of shapes, we generate 1000 samples of given size. For each sample we estimate $e(x)$ according to our new procedures in section 2. In addition, we also estimate $e(x)$ using the Yang's (1978) estimators. The Bias, Variance(VAR), and Mean Squared Error(MSE) of our estimators are compared with those of the Yang's (1978) estimator given distributions at each deciles of distribution F.

Table 4.1-4.5 indicate the result of simulations with varying shape parameter, and sample size n=100. From Table 4.1-4.5 we notice that our new estimator seems to produce less MSE than the MSE of Yang's (1978) empirical estimator.

Table 4.1 Results of simulation from weibull distribution with parameter a=0.7.

$F(x)$	Empirical			Proposed			Ratio of MSE
	BIAS	VAR	MSE	BIAS	VAR	MSE	
.1	-.0026	.0419	.0419	-.0030	.0417	.0418	1.0024
.2	-.0013	.0503	.0503	-.0016	.0501	.0501	1.0040
.3	-.0003	.0636	.0636	-.0005	.0635	.0635	1.0016
.4	.0005	.0777	.0777	.0000	.0774	.0774	1.0039
.5	-.0040	.1020	.1020	-.0044	.1015	.1016	1.0039
.6	-.0047	.1333	.1333	-.0055	.1317	.1318	1.0114
.7	-.0026	.1971	.1971	-.0058	.1943	.1943	1.0144
.8	-.0088	.3275	.3276	-.0114	.3267	.3268	1.0024
.9	-.0360	.7875	.7888	-.0552	.7320	.7351	1.0731

Table 4.2 Results of simulation from weibull distribution with parameter a=1.

$F(x)$	Empirical			Proposed			Ratio of MSE
	BIAS	VAR	MSE	BIAS	VAR	MSE	
.1	-.0013	.0115	.0115	-.0015	.0115	.0115	1.0000
.2	.0000	.0123	.0123	.0000	.0122	.0122	1.0082
.3	.0009	.0142	.0142	.0008	.0140	.0140	1.0143
.4	-.0007	.0163	.0163	-.0010	.0162	.0162	1.0062
.5	.0002	.0201	.0201	-.0001	.0199	.0199	1.0101
.6	-.0004	.0240	.0240	-.0005	.0235	.0235	1.0213
.7	-.0041	.0335	.0335	-.0062	.0328	.0329	1.0182
.8	-.0140	.0469	.0471	-.0161	.0458	.0460	1.0239
.9	-.0189	.0973	.0977	-.0266	.0906	.0914	1.0689

Table 4.3 Results of simulation from weibull distribution with parameter a=2.

$F(x)$	Empirical			Proposed			Ratio of MSE
	BIAS	VAR	MSE	BIAS	VAR	MSE	
.1	-.0007	.0019	.0019	-.0003	.0018	.0018	1.0556
.2	-.0017	.0020	.0020	-.0012	.0019	.0019	1.0526
.3	-.0016	.0020	.0020	-.0010	.0019	.0019	1.0526
.4	-.0005	.0020	.0020	.0003	.0020	.0020	1.0000
.5	-.0010	.0021	.0021	-.0001	.0021	.0021	1.0000
.6	-.0008	.0024	.0024	.0001	.0023	.0023	1.0435
.7	-.0018	.0029	.0029	-.0007	.0028	.0028	1.0357
.8	-.0008	.0040	.0040	.0007	.0039	.0039	1.0256
.9	-.0010	.0076	.0076	.0016	.0066	.0066	1.1515

Table 4.4 Results of simulation from gamma distribution with parameter a=2.

$F(x)$	Empirical			Proposed			Ratio of MSE
	BIAS	VAR	MSE	BIAS	VAR	MSE	
.1	.0034	.0194	.0194	.0042	.0193	.0193	1.0029
.2	.0037	.0209	.0209	.0039	.0208	.0208	1.0060
.3	.0040	.0235	.0235	.0045	.0234	.0234	1.0051
.4	-.0012	.0267	.0267	-.0007	.0263	.0263	1.0140
.5	.0029	.0307	.0308	.0037	.0304	.0304	1.0128
.6	.0038	.0386	.0386	.0058	.0378	.0378	1.0218
.7	.0047	.0521	.0521	.0056	.0507	.0507	1.0274
.8	.0121	.0759	.0760	.0133	.0747	.0749	1.0151
.9	.0313	.1713	.1723	.0245	.1543	.1549	1.1123

Table 4.5 Results of simulation from gamma distribution with parameter a=3.

$F(x)$	Empirical			Proposed			Ratio
	BIAS	VAR	MSE	BIAS	VAR	MSE	of MSE
.1	.0015	.0298	.0298	.0035	.0295	.0295	1.0090
.2	.0011	.0310	.0310	.0033	.0306	.0306	1.0124
.3	.0005	.0333	.0333	.0026	.0330	.0330	1.0115
.4	-.0016	.0371	.0371	.0013	.0366	.0366	1.0129
.5	.0015	.0426	.0426	.0046	.0420	.0421	1.0130
.6	.0012	.0509	.0509	.0055	.0499	.0500	1.0188
.7	.0036	.0659	.0659	.0086	.0642	.0642	1.0265
.8	-.0020	.0935	.0935	.0059	.0904	.0904	1.0338
.9	-.0128	.1905	.1906	-.0023	.1755	.1755	1.0863

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