A FAMILY OF TESTS FOR TREND CHANGE IN MEAN RESIDUAL LIFE WITH UNKNOWN TURNING POINT

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ABSTRACT

A family of test statistics is proposed for testing whether or not the mean residual life(MRL) changes its trend. We do not assume that the turning point or the proportion before the turning point is known. This family includes the test statistic proposed by Aly (1990) and Hawkins, Kochar and Loader (1992) for complete samples. We establish the asymptotic null distribution of test statistics and obtain asymptotic critical values of the asymptotic null distribution using Durbin's approximation. We study Monte Carlo simulation to compare the proposed tests with previously known tests.

1. Introduction

Let X denote the lifetime of an item having a continuous distribution function F such that $F(0) = 0$ and let $\bar{F}(x) = 1 - F(x)$. The mean residual life(MRL) function, $e(x)$, is defined by

$$e(x) = \begin{cases} \int_x^\infty \bar{F}(t)dt/\bar{F}(x), & \text{if } \bar{F}(x) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The MRL is the expected remaining lifetime, $X - x$, given that the item has survived to time x. The unconditional mean of the distribution, $\mathrm{E}(X)$, is a special case given by $e(0)$.

The MRL function plays a very important role in many areas, including engineering, medical science, and actuarial sciences. The MRL is used by engineers in burn-in studies, setting maintenance policies, and in comparison of life distributions of different systems. Medical researchers use MRL in lifetime experiments under various conditions. Actuaries apply MRL to setting rates and benefits for life insurance. See Hall and Wellner (1981), and Guess and Proschan (1988) for more detail of theory and application of MRL.

Based on the behavior of MRL function, various nonparametric classes of life distributions have been defined using MRL. One such class consists of those with "increasing then decreasing mean residual life (IDMRL)". F is called an IDMRL distribution if there exists a turning point $\tau \geq 0$ such that

$$\begin{aligned} e(s) &\leq e(t), \quad \text{for } 0 \leq s \leq t < \tau, \\ e(s) &\geq e(t), \quad \text{for } \tau \leq s \leq t. \end{aligned}$$

See Guess and Proschan(1988) and the references therein for examples and applications of the IDMRL class. Also it is well known that F is exponential distribution if and only if $e(x)$ is constant. In this paper we consider the problem of testing

H_0 : F is the exponential distribution

(i.e. $\bar{F}(x) = \exp(-x/\mu)$, $x \geq 0$, with μ unknown)

against

H_1 : F is IDMRL, but not exponential.

based on a random sample $X_1, \cdots, X_n$ from a continuous distribution F. Guess, Hollander and Proschan (1986) propose two test procedures for constant MRL against the trend change in MRL when the turining point τ is known or when the proportion $p = F(\tau)$ before the change occurs is known. Aly (1990) suggests several tests for monotonicity of MRL. These tests consider the IDMRL alternative when neither the change point nor the proportion is known. Also Hawkins, Kochar and Loader(henceforth HKL, 1992) develope a test for exponentiality against IDMRL alternative when neither the change point nor the proportion is known. Recently, Lim and Park (1998) study a family of IDMRL tests when the proportion is known.

In section 2 we develope a family of test statistics for testing H_0 against H_1 without assuming that the turning point or the proportion is known. This family includes the test statistics proposed by Aly (1990) and HKL (1992) as its special cases. We derive the aymptotic null distributions of our test statistics. In section 3 Monte Carlo simulations are conducted to investigate the speed of convergence of the proposed family of test statistics to the asymptotic null distribution, and to compare the performance of our test statistics with those of Aly's (1990) and HKL's (1992) tests by the power of tests.

2. The Family of IDMRL Tests

Our test statistics are motivated by a simple observation which is a idea of Ahmad (1992). If $e(x)$ is differentiable and $e(x)$ is decreasing(increasing), then

$$e'(x) = \frac{de(x)}{dx} = \frac{f(x)v(x) - \bar{F}^2(x)}{\bar{F}^2(x)} \le (\ge) 0,$$

where $v(x) = \int_x^\infty \bar{F}(u)du$ and f denotes the probability density function corresponding to F. Thus $e(x)$ is nondecreasing(nonincreasing) if and only if $f(x)v(x) \le (\ge) \bar{F}^2(x)$. Hence, as a measure of the deviation from the null hypothesis H_0 in favor of H_1 we propose the parameter

$$T_j(F) = \sup\{\phi_j(x; F) : x \ge 0\}$$

where

$$\begin{aligned} \phi_j(x;F) &= \int_0^x \bar{F}^j(t)(f(t)v(t) - \bar{F}^2(t))dt \\ &\quad + \int_x^\infty \bar{F}^j(t)(\bar{F}^2(t) - f(t)v(t))dt \qquad (2.1) \end{aligned}$$

where j is a integer with $j \ge -1$. This parameter (2.1) coincides with that of Aly (1990) and HKL (1992) when $j = -1$ and $j = 0$, respectively. Note that $\phi_j(x; F)$ is differentiable in $x > 0$ and

$$\frac{d}{dx}\phi_j(x;F) = 2\bar{F}^{j+2}(x)e'(x),$$

clearly has the same sign as does $e'(x)$. Thus, if F is IDMRL, but not exponential, then $\phi_j(x;F)$ is strictly increasing (decreasing) for $x < \tau$ ($x > \tau$) and $T_j(F) = \phi_j(\tau;F) > 0$. If F is exponential, then $e(x)$ is constant and $e'(x) = 0$ for all $x \geq 0$, and hence the integrand of $\phi_j(x;F)$ is zero, so that $T_j(F) = 0$. Using integration by parts, we can rewrite $\phi_j(x;F)$ as

$$\begin{aligned}\phi_j(x;F) &= \frac{1}{j+1}\bigg(\int_0^\infty \bar{F}(t)dt - (j+2)\int_0^x \bar{F}^{j+2}(t)dt \\ &\quad +(j+2)\int_x^\infty \bar{F}^{j+2}(t)dt - 2\bar{F}^{j+1}(x)\int_x^\infty \bar{F}(t)dt\bigg).\end{aligned}$$

Let $F_n(x)$ denote the empirical distribution, and let $\bar{X}$ denote the sample mean. Then our family of test statistics is

$$T_j^* = \frac{\sqrt{n}T_j(F_n)}{\bar{X}}.$$

For computational purpose, T_j^* may be written as

$$T_j^* \simeq \frac{\max\limits_{0\leq k\leq n}\sqrt{n}(2\eta_j(k) - \eta_j(0))}{\bar{X}}$$

where for $k = 0, 1, \cdots, n$

$$\eta_j(k) = \frac{1}{j+1}\sum_{i=k}^{n-1}\left\{(j+2)\left(\frac{n-i}{n}\right)^{j+2} - \left(\frac{n-k}{n}\right)^{j+1}\left(\frac{n-i}{n}\right)\right\}(X_{(i+1)} - X_{(i)}),$$

$0 = X_{(0)} < X_{(1)} < \cdots < X_{(n)}$ denote the order statistics of the sample. For $j \to -1$ and $j = 0$, our test statistics are equal to the test statistics of Aly (1990) and HKL (1992), respectively.

To establish the asymptotic null distribution of T_j^*, we use the differentiable statistical function approach of von Mises (1947) (cf. Boos and Serfling (1980) and Serfling (1980)) and the classical weak convergence of the empirical process. Also see HKL (1992) that obtain the asymptotic null distribution of a test statistic T_0^*. The asymptotic null distribution of T_j^* is summarized in Theorem 2.1.

THEOREM 2.1 Under H_0, i.e. F is exponential distribution with mean μ,

$$T_j^* \xrightarrow{\mathcal{L}} Z_j^* \equiv \sup\{Z_j(p) : 0 \leq p \leq 1\},$$

Table 2.1 Approximated quantiles of T_j^*.

α	quantiles of T_j^*					
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
0.90	1.406	1.089	0.921	0.812	0.734	0.676
0.95	1.588	1.230	1.039	0.917	0.829	0.763
0.99	1.930	1.495	1.263	1.114	1.008	0.927

where $Z_j(p)$ denote a mean zero Gaussian process with covariance

$$\sigma_j(p,q) = \frac{1}{2j+3}(1 + 2(1-q)^{2j+3} - 2(1-p)^{2j+3}) \quad \text{for} \quad p \le q.$$

Using Durbin's (1985) approximation, we can obtain asymptotic critical values based on the distribution of T_j^* by

$$\Pr\{T_j^* > c\} = \{2\sqrt{2j+3}c + O(c^{-1})\}\phi(\sqrt{2j+3}c) \quad \text{as} \quad c \to \infty \tag{2.2}$$

where ϕ denote the probability density function of the standard normal distribution. Table 2.1 contains approximated quantiles of the distribution of T_j^* for some different j, computed from (2.2).

3. Simulation Study

In this section we perform a Monte Carlo simulation to investigate the speed of convergence of the proposed test statistics and the performance of our test statistics by simulating the power of tests. Simulations are performed on a super computer SP2 at Seoul National University using the programing language FORTRAN.

To investigate the empirical test size, the random numbers are generated from exponential distribution, $F(x) = 1 - \exp(-x)$, $x \ge 0$, since our test statistics are scale invariant. Table 3.1 presents the empirical test size of IDMRL tests based on T_j^* for some different j.

The values in Tables 3.1 are the fraction of times that H_0 is rejected in favor of H_1 when H_0 is true. The empirical test sizes are calculated based on 1000 replications for; α =0.10, 0.05, 0.01; $n = 10, 20, \cdots, 100$. From Table 3.1, we notice that the fastest convergence of T_j^* is obtained by using T_1^*. The test size of T_1^* is close to the level of significance when $n \geq 30$. The T_{-1}^* test overestimate α. The T_j^* test, for the large value of j, slightly underestimate α.

To evaluate the empirical power of the proposed tests, the random numbers are generated from

$$\begin{aligned}\bar{F}_{\alpha,\beta,\gamma}(x) &= \left\{\frac{\beta}{\beta+\gamma\exp(-\alpha x)(1-\exp(-\alpha x))}\right\}\left\{\frac{[1+d]^2-c^2}{[\exp(\alpha x)+d]^2-c^2}\right\}^{1/2\alpha\beta} \\ &\quad\times\left\{\frac{[\exp(\alpha x)+d-c][1+d+c]}{[\exp(\alpha x)+d+c][1+d-c]}\right\}^{\gamma/4\alpha\beta^2 c}, \quad x \geq 0, \alpha > 0, \beta > 0, \gamma > 0\end{aligned}$$

where $d = \gamma/2\beta$, $c^2 = (4\beta\gamma + \gamma^2)/(4\beta^2)$. This distribution has MRL function $e_{\alpha,\beta,\gamma}(x) = \beta + \gamma\exp(-\alpha x)(1-\exp(-\alpha x))$, $x \geq 0$. The motivation (see, HKL 1992) for choosing $\bar{F}_{\alpha,\beta,\gamma}$ is that $\bar{F}_{\alpha,\beta,\gamma}$ has IDMRL structure with the turning point $\tau = \ln 2/\alpha$ for any choice of (α, β, γ) and $\bar{F}_{\alpha,\beta,\gamma}$ is exponential distribution if $\gamma = 0$.

Tables 3.2–3.4 present the empirical powers of our family of IDMRL tests based on T_j^* for some different j when testing against alternative $\bar{F}_{\alpha,\beta,\gamma}$, where the random numbers are generated for various choice of α with $\beta = 1$ and $\gamma = 1$. The values in Tables are based on 1000 replications for various sample size n.

From Tables 3.2–3.4, we notice that the power of all T_j^* tests increase rapidly as α increases (i.e., the turning point τ decreases) when β and γ is fixed. When the parameter $\alpha = 5$, the largest power of T_j^* tests is obtained by using $j = 1$. The largest power of T_j^* is obtained by using $j = 0$ when $\alpha = 3$. When the parameter $\alpha = 1$, the largest power of T_j^* is obtained by using T_{-1}^*.

Table 3.1 *Empirical test size of IDMRL tests based on T_j^* for some j.*

n	α	j=-1	j=0	j=1	j=2	j=3	j=4	j=5
	.10	.106	.097	.084	.087	.098	.102	.116
10	.05	.062	.049	.041	.047	.058	.071	.079
	.01	.025	.011	.009	.018	.024	.030	.037
	.10	.139	.112	.085	.071	.074	.075	.088
20	.05	.069	.052	.036	.038	.038	.038	.042
	.01	.024	.008	.011	.012	.012	.016	.019
	.10	.121	.119	.105	.095	.093	.091	.096
30	.05	.067	.062	.049	.044	.050	.052	.061
	.01	.017	.007	.015	.019	.020	.019	.019
	.10	.148	.120	.110	.103	.103	.097	.088
40	.05	.095	.061	.052	.056	.053	.052	.057
	.01	.029	.013	.011	.014	.017	.017	.015
	.10	.136	.106	.092	.081	.086	.087	.079
50	.05	.076	.045	.044	.041	.039	.039	.043
	.01	.025	.006	.008	.010	.010	.013	.014
	.10	.145	.117	.105	.089	.084	.071	.072
60	.05	.070	.053	.046	.044	.040	.041	.040
	.01	.015	.004	.005	.008	.010	.011	.012
	.10	.148	.115	.096	.100	.088	.086	.084
70	.05	.068	.048	.048	.048	.049	.047	.043
	.01	.019	.009	.006	.009	.012	.014	.015
	.10	.153	.126	.106	.092	.078	.078	.079
80	.05	.090	.060	.053	.040	.041	.042	.041
	.01	.024	.014	.011	.011	.015	.015	.019
	.10	.125	.099	.098	.097	.099	.098	.094
90	.05	.058	.046	.041	.039	.044	.047	.048
	.01	.013	.008	.006	.012	.011	.010	.012
	.10	.142	.123	.098	.093	.092	.086	.086
100	.05	.072	.054	.047	.049	.046	.041	.039
	.01	.017	.006	.007	.010	.012	.013	.014

Table 3.2 *Empirical power of* T_j^* *when testing against alternative* $\bar{F}_{\alpha,\beta,\gamma}$ *with parameter* $\alpha = 5$, $\beta = 1$ *and* $\gamma = 1$.

n	α	$j=-1$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
	.10	.600	.751	.772	.699	.553	.345	.163
20	.05	.455	.629	.593	.476	.272	.106	.058
	.01	.226	.351	.266	.109	.039	.022	.016
	.10	.830	.948	.962	.952	.942	.906	.835
40	.05	.738	.894	.911	.908	.854	.770	.594
	.01	.461	.735	.740	.664	.475	.248	.060
	.10	.924	.985	.994	.995	.992	.982	.969
60	.05	.856	.967	.981	.977	.966	.948	.929
	.01	.627	.878	.908	.903	.845	.741	.557
	.10	.957	.996	.999	1.000	1.000	1.000	.998
80	.05	.919	.990	.996	.997	.998	.995	.993
	.01	.750	.964	.980	.979	.966	.948	.900
	.10	.985	1.000	1.000	1.000	1.000	1.000	1.000
100	.05	.964	.998	1.000	1.000	1.000	.999	.999
	.01	.870	.993	.998	.998	.998	.995	.985

Table 3.3 *Empirical power of* T_j^* *when testing against alternative* $\bar{F}_{\alpha,\beta,\gamma}$ *with parameter* $\alpha = 3$, $\beta = 1$ *and* $\gamma = 1$.

n	α	$j=-1$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
	.10	.603	.690	.590	.448	.250	.101	.039
20	.05	.463	.533	.415	.235	.079	.027	.016
	.01	.224	.275	.126	.032	.013	.006	.004
	.10	.805	.900	.897	.842	.750	.617	.477
40	.05	.681	.811	.775	.688	.524	.376	.218
	.01	.447	.577	.484	.330	.155	.029	.003
	.10	.905	.971	.976	.962	.936	.877	.813
60	.05	.825	.938	.940	.903	.824	.735	.612
	.01	.610	.790	.754	.659	.495	.283	.131
	.10	.951	.991	.987	.983	.977	.962	.945
80	.05	.890	.973	.972	.961	.941	.909	.851
	.01	.733	.892	.897	.849	.773	.653	.505
	.10	.966	.995	.998	.998	.995	.990	.984
100	.05	.943	.989	.993	.991	.982	.973	.952
	.01	.822	.953	.960	.945	.915	.856	.748

Table 3.4 *Empirical power of T_j^* when testing against alternative $\bar{F}_{\alpha,\beta,\gamma}$ with parameter $\alpha = 1$, $\beta = 1$ and $\gamma = 1$.*

n	α	$j=-1$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
	.10	.491	.361	.191	.081	.032	.013	.008
20	.05	.354	.233	.090	.024	.006	.002	.002
	.01	.178	.083	.015	.002	.001	.001	.001
	.10	.636	.572	.417	.270	.167	.092	.042
40	.05	.513	.422	.265	.133	.052	.019	.004
	.01	.289	.191	.064	.012	.001	.000	.000
	.10	.764	.694	.574	.424	.312	.232	.148
60	.05	.635	.557	.399	.269	.170	.098	.050
	.01	.391	.299	.158	.058	.027	.009	.002
	.10	.843	.815	.726	.614	.510	.412	.322
80	.05	.750	.711	.578	.465	.341	.240	.162
	.01	.509	.459	.300	.185	.088	.033	.015
	.10	.894	.886	.825	.742	.642	.541	.449
100	.05	.821	.794	.695	.578	.467	.360	.276
	.01	.589	.577	.438	.294	.178	.094	.044

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