Generalization of Staggered Nested Designs for Precision Experiments

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Abstract

Staggered nested designs are the most popular class of unbalanced nested designs in practical fields. The most important features of the staggered nested design are that it has a very simple open-ended structure and each sum of squares in the analysis of variance has almost the same degrees of freedom. Based on the features, a class of unbalanced nested designs which is generalized of the staggered nested design is proposed. Some of the generalized staggered nested designs are shown to be more efficient than the staggered nested design in estimating some of variance components and their linear combinations.

1. Introduction

The estimation of variance components is applied in many practical fields including not only the quality improvements of most processes but also standardizing measurement methods. For standardizing measurement methods, precision experiments are required to evaluate their precision data, which are calculated from variance components estimates. Processes are usually divided into many steps, and estimates of the variance components of the steps are quite useful to identify major sources of output variation. The balanced nested design is usually used for this purpose, owing to its easiness in both administration and statistical analysis. However the balanced design has a defect in having relatively less degrees of freedom for the factors at the upper parts of the hierarchy. To eliminate this defect, several unbalanced nested designs have been proposed.

The staggered nested design which was proposed and named by Bainbridge[2)], is the most popular unbalanced nested design in practical fields, and is recommended in Annex C of ISO 5725-3[4)]. An experimental unit of the experiment is shown as Design 1 in Figure 1. A set of the observations under a lot (uppermost factor in the hierarchy) is referred as an experimental unit in this paper. The staggered nested design has three important features. They are: (a) an open-ended structure, (b) having almost same and least degrees of freedom, and (c) the sums of squares having the χ^2 type distributions. The staggered nested design is made up of n times replication of identical experimental units. Bainbridge[2)] referred such design as an open-ended nested design. This feature is satisfactory for the administration of the experiments in practical fields. The analysis of variance (ANOVA) is usually carried out, because the total sum of squares is uniquely decomposed into sums of squares corresponding factors for the analysis of unbalanced nested designs. When the number of replication of the experimental units is n, the degrees of the freedom for the uppermost factor in the hierarchy is n-1, and the degrees of the freedoms for the other factors are all n. Ojima[9] showed that under the assumption of the normality for all random effects and the independence of the experimental units, all of the sums of squares distribute as a constant times of χ^2 distributions. The constant is equal to the expectation of the mean square. Based on the features mentioned above, we generalize the staggered nested design in Section 2.

The multi-stage (five or more-stage) nested design is also useful for the case of application for the standardization of chemical analyses. There are many factors which may affect test results as measurement errors, should be investigated in the process of practical chemical analyses. The large-scale multi-stage nested experiment can be workable for the case of international standardization of measurement methods, because the experiment provides much information for many factors at one time.

There are two four-stage generalized staggered nested designs (shown in Figure 1) including the traditional staggered nested design. They are discussed and compared in Section 3. The new design is shown to be more efficient in estimating some of the variance components. Five five-stage generalized staggered nested designs are shown in Figure 2. Some of the generalized staggered nested designs appeared in the past articles. Calvin and Miller[3] introduced a four-stage design (Design 2 in Figure 1), and presented the expected mean squares for the analysis of variance for the design. Anderson[1] referred the same design and a five-stage design (Design 4 in Figure 2) from R. R. Prairie's unpublished Ph. D. Thesis.

Many authors addressed the associated problem for the staggered nested design. Bainbridge[2] presented a procedure of the analysis of variance and the expected mean squares for three to six-stage designs. Nelson[6),7] illustrated the point estimation procedure and approximate confidence limits of variance components. Ojima[8] applied the control chart method for analyzing staggered nested data. Khattree and Naik[5] proposed a statistical test for variance components in the staggered nested designs. Ojima[9] presented general formulas for the expectations, the variances and the covariances of the mean squares for the staggered nested design.

2. Generalized staggered nested design

Based on the features mentioned in Section 1, we generalize the staggered nested design. From the feature of (a) an open-ended structure, it is necessary and sufficient that the design be made up of n times replication of identical experimental units. The feature (b) having almost same and least degrees of freedom, requires that one branch generated in each stage in the experimental unit. Applying the orthogonal transformation in the experimental unit as described in Ojima[9], the sums of squares obviously distributed as the χ^2 type distributions under the assumption of the normality for all random effects and the independence of the experimental units.

There is only one three-stage nested design satisfies the above condition, and is the traditional staggered nested design. But there are two four-stage generalized staggered nested designs which are shown in Figure 1. Design 1 in Figure 1 is the staggered nested design. The five-stage generalized staggered nested designs are shown in Figure 2.

The number of the generalized staggered nested designs can be calculated as the following manner. For the case of the four-stage nested designs (Figure 1), Design 1 is characterized as having a paired observations (y_{i1} and y_{i2}) and two unpaired observations (y_{i3} and y_{i4}). Similarly Design 2 is described as having two sets paired observations i.e. (y_{i1} and y_{i2}) and (y_{i3} and y_{i4}). Let symbol (p, q) be a design having p sets of paired observations and q unpaired observations. Obviously, $p \geq 1$ and $q \geq 0$. So Design 1 is (1, 2) and Design is (2, 0). For the case of the five-stage nested designs (Figure 2), Design 1 is (1, 3) and the others are all (2, 1). Let $n(p, q)$ be the number of designs (p, q). Then $n(1, 3)$=1 and $n(2, 1)$=4.

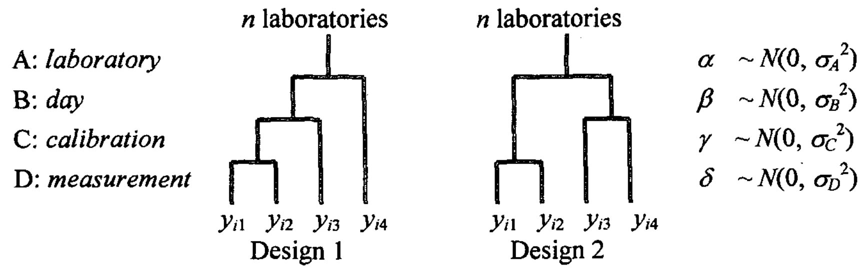

Figure 1 Experimental unit of Four-stage Generalized Staggered Nested Designs

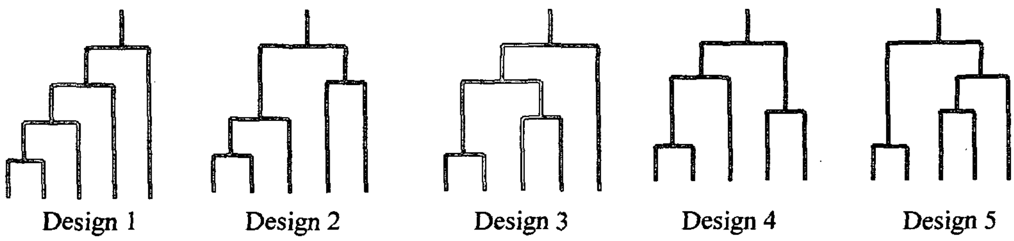

Figure 2 Experimental unit of Five-stage Generalized Staggered Nested Designs

To generate $(k+1)$-stage designs, one branch should be put on one of the observations of a k-stage design. The paired observations generate the same $(k+1)$-design, but unpaired observations generate different designs. For the case of Design 1 in Figure 1, putting a branch on y_{i1} or y_{i2} makes the same Design 1 in Figure 2, but putting a branch on y_{i3} makes Design 3 in Figure 2, and putting a branch on y_{i4} makes Design 5 in Figure 2. Similarly for the case of Design 2 in Figure 1, putting a branch on y_{i1} or y_{i2} makes the same Design 2 in Figure 2, and putting a branch on y_{i3} or y_{i4} makes Design 4 in Figure 2. So (p, q) design generates p types of $(p, q+1)$ design and q types of $(p+1, q-1)$ design. Further the different k-stage designs generate obviously different $(k+1)$-stage designs. Then we have a recurrence formula,

$$n(p, q) = (q+1)\cdot n(p-1, q+1) + p\cdot n(p, q-1). \qquad (2.1)$$

For the k-stage designs, let $N(k)$ be the number of k-stage designs, $N(k)$ is obtained as

$$N(k)=n(1, k-2) + n(2, k-4)+ \cdots +n(m, k-2m), \qquad (2.2)$$

where $m = [k/2]$ is an integer part of $k/2$. Obviously we have $N(3) = n(1, 1) = 1$. Applying the recurrence formula, we obtain $N(4) = 2$, $N(5) = 5$, $N(6) = 16$, $N(7) = 61$, $N(8) = 272$, $N(9) = 1385$, $N(10) = 7936$, ... , $N(15) = 199360981$, ... , $N(20) = 29088885112832$, ...

3. Four-stage generalized staggered nested designs

3.1 Analysis of variance

Figure 1 shows an example of the experimental units of the four-stage generalized staggered nested design (Design 2) and the Staggered Nested Design (Design 1). The purpose of the experiment is to estimate four variance components. They are σ_A^2: variance between laboratories, σ_B^2: variance between days, σ_C^2: variance of calibration effects, and σ_D^2: variance of measurement errors. They are used to evaluate the reproducibility variance, $\sigma_R^2 = \sigma_A^2 + \sigma_B^2 + \sigma_C^2 + \sigma_D^2$, and the repeatability variance, $\sigma_r^2 = \sigma_D^2$, which are the most important precision measures of measurement methods and indispensable to standardize a measurement method. The intermediate precision measures, $\sigma_B^2 + \sigma_C^2 + \sigma_D^2$ and $\sigma_C^2 + \sigma_D^2$ are also important for internal quality control of measurement processes. Analysis of variance is usually used for the variance components estimation. Let y_{ij} be j-th observation at i-th laboratory as shown in Figure 1. The sums of squares are calculated as the following manner.

For the Design 1 (Staggered nested design):

$$SSA = \Sigma_i (\Sigma_j y_{ij})^2/4 - CT, \text{ where } CT = (\Sigma_i \Sigma_j y_{ij})^2/(4n), \quad SSB = \Sigma_i (y_{i1} + y_{i2} + y_{i3} - 3y_{i4})^2/12,$$
$$SSC = \Sigma_i (y_{i1} + y_{i2} - 2y_{i3})^2/6, \quad SSD = \Sigma_i (y_{i1} - y_{i2})^2/2. \qquad (3.1)$$

And for the Design 2:

$$SSA = \Sigma_i (\Sigma_j y_{ij})^2/4 - CT, \text{ where } CT = (\Sigma_i \Sigma_j y_{ij})^2/(4n), \quad SSB = \Sigma_i (y_{i1} + y_{i2} - y_{i3} - y_{i4})^2/4,$$
$$SSC = \Sigma_i (y_{i3} - y_{i4})^2/2, \quad SSD = \Sigma_i (y_{i1} - y_{i2})^2/2. \qquad (3.2)$$

The mean squares are obtained as the usual manner, i.e. $MSA = SSA/(n-1)$, $MSB = SSB/n$, $MSC = SSC/n$, and $MSD = SSD/n$. The expectations of the mean squares are listed in Tables 1 and 2.

Table 1 ANOVA Table for Design 1 (four-stage staggered nested design)

Source	*d.f.*	*E(m. s.)*
A	*n*-1	$\sigma_D^2 + \frac{3}{2}\sigma_C^2 + \frac{5}{2}\sigma_B^2 + 4\sigma_A^2$
B	*n*	$\sigma_D^2 + \frac{7}{6}\sigma_C^2 + \frac{3}{2}\sigma_B^2$
C	*n*	$\sigma_D^2 + \frac{4}{3}\sigma_C^2$
D	*n*	σ_D^2

Table 2 ANOVA Table for Design 2

Source	*d.f.*	*E(m. s.)*
A	*n*-1	$\sigma_D^2 + \frac{3}{2}\sigma_C^2 + 2\sigma_B^2 + 4\sigma_A^2$
B	*n*	$\sigma_D^2 + \frac{3}{2}\sigma_C^2 + 2\sigma_B^2$
C	*n*	$\sigma_D^2 + \sigma_C^2$
D	*n*	σ_D^2

Variance components are estimated by the equating mean squares with their expectations. The estimators are unbiased, and usually referred as ANOVA estimators. The ANOVA estimators are

$$\hat{\sigma}_A^2 = (3MSA - 5MSB + MSC + MSD)/12, \quad \hat{\sigma}_B^2 = (8MSB - 7MSC + MSD)/12,$$
$$\hat{\sigma}_C^2 = 3(MSC - MSD)/4, \quad \hat{\sigma}_D^2 = MSD, \qquad \text{by Design 1} \qquad (3.3)$$
$$\hat{\sigma}_A^2 = (MSA - MSB)/4, \quad \hat{\sigma}_B^2 = (2MSB - 3MSC + MSD)/4,$$
$$\hat{\sigma}_C^2 = MSC - MSD, \quad \hat{\sigma}_D^2 = MSD. \qquad \text{by Design 2} \qquad (3.4)$$

3.2 Comparison of four-stage generalized staggered nested designs

Variances of these estimators are shown in Appendix, they are obtained under the normality assumption of all random effects. The results of the comparisons of estimators based on their variances are as follows.

(1) Estimation of variance components

a) uniformly var($\hat{\sigma}_A^2$| Design 1) > var($\hat{\sigma}_A^2$| Design 2); i.e. Design 2 always gives precise estimates.

b) practically var($\hat{\sigma}_B^2$| Design 1) > var($\hat{\sigma}_B^2$| Design 2); i.e. the case that Design 2 gives precise estimates is depend on the true value of σ_B^2, σ_C^2, and σ_D^2. If $\sigma_B^2 > 0.25\sigma_D$, or $\sigma_C^2 > 0.81\sigma_D^2$ that is thought to cover most practical cases, var($\hat{\sigma}_B^2$| Design 1) > var($\hat{\sigma}_B^2$| Design 2) is always established.

c) uniformly var($\hat{\sigma}_C^2$| Design 1) < var($\hat{\sigma}_C^2$| Design 2); i.e. Design 1 always gives precise estimates.

d) always var($\hat{\sigma}_D^2$| Design 1) = var($\hat{\sigma}_D^2$| Design 2); i.e. estimates have the same precision.

(2) Estimation of sum of variance components (precision measures)

a) For the reproducibility variances, uniformly var($\hat{\sigma}_R^2$| Design 1) > var($\hat{\sigma}_R^2$| Design 2).

b) For the intermediate precision measures, uniformly var($\hat{\sigma}_B^2+\hat{\sigma}_C^2+\hat{\sigma}_D^2$| Design 1) > var($\hat{\sigma}_B^2+\hat{\sigma}_C^2+\hat{\sigma}_D^2$| Design 2), but uniformly var($\hat{\sigma}_C^2+\hat{\sigma}_D^2$| Design 1) < var($\hat{\sigma}_C^2+\hat{\sigma}_D^2$| Design 2).

4. Conclusion

Based on the features of the staggered nested design, a class of unbalanced nested designs is proposed. For the four-stage generalized staggered nested designs, the variances of the estimators are compared. The generalized staggered nested designs are shown to be more efficient than the conventional staggered nested design in estimating some of variance components and their linear combinations.

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***Keywords:* variance component, analysis of variance**

Appendix: Variances and covariances of the estimators

For Design 1

$$\mathrm{var}\left(\hat{\sigma}_A^{\,2}\right)=\tfrac{2}{n}\left(\tfrac{3}{16}\sigma_D^{\,4}+\tfrac{15}{64}\sigma_C^{\,4}+\tfrac{61}{144}\sigma_C^{\,2}\sigma_D^{\,2}+\tfrac{15}{64}\sigma_B^{\,4}+\tfrac{25}{48}\sigma_B^{\,2}\sigma_D^{\,2}+\tfrac{145}{288}\sigma_B^{\,2}\sigma_C^{\,2}\right)$$
$$+\tfrac{2}{n-1}\left(\tfrac{1}{4}\sigma_D^{\,2}+\tfrac{3}{8}\sigma_C^{\,2}+\tfrac{5}{8}\sigma_B^{\,2}+\sigma_A^{\,2}\right)^2$$
$$\mathrm{var}\left(\hat{\sigma}_B^{\,2}\right)=\tfrac{2}{n}\left(\tfrac{19}{24}\sigma_D^{\,4}+\tfrac{7}{6}\sigma_C^{\,4}+\tfrac{35}{18}\sigma_C^{\,2}\sigma_D^{\,2}+\sigma_B^{\,4}+\tfrac{4}{3}\sigma_B^{\,2}\sigma_D^{\,2}+\tfrac{14}{9}\sigma_B^{\,2}\sigma_C^{\,2}\right)$$
$$\mathrm{var}\left(\hat{\sigma}_C^{\,2}\right)=\tfrac{2}{n}\left(\tfrac{9}{8}\sigma_D^{\,4}+\sigma_C^{\,4}+\tfrac{3}{2}\sigma_C^{\,2}\sigma_D^{\,2}\right),\quad \mathrm{var}\left(\hat{\sigma}_D^{\,2}\right)=\tfrac{2}{n}\sigma_D^{\,4}$$
$$\mathrm{cov}\left(\hat{\sigma}_A^{\,2},\hat{\sigma}_B^{\,2}\right)=-\tfrac{2}{n}\left(\tfrac{1}{3}\sigma_D^{\,4}+\tfrac{11}{24}\sigma_C^{\,4}+\tfrac{7}{9}\sigma_C^{\,2}\sigma_D^{\,2}+\tfrac{1}{2}\sigma_B^{\,4}+\tfrac{5}{6}\sigma_B^{\,2}\sigma_D^{\,2}+\tfrac{8}{9}\sigma_B^{\,2}\sigma_C^{\,2}\right)$$
$$\mathrm{cov}\left(\hat{\sigma}_A^{\,2},\hat{\sigma}_C^{\,2}\right)=\tfrac{1}{4n}\sigma_C^{\,4}+\tfrac{1}{3n}\sigma_C^{\,2}\sigma_D^{\,2},\quad \mathrm{cov}\left(\hat{\sigma}_B^{\,2},\hat{\sigma}_C^{\,2}\right)=-\tfrac{2}{n}\left(\tfrac{3}{8}\sigma_D^{\,4}+\tfrac{3}{4}\sigma_C^{\,4}+\tfrac{7}{6}\sigma_C^{\,2}\sigma_D^{\,2}\right)$$
$$\mathrm{cov}\left(\hat{\sigma}_A^{\,2},\hat{\sigma}_D^{\,2}\right)=\tfrac{1}{6n}\sigma_D^{\,4},\quad \mathrm{cov}\left(\hat{\sigma}_B^{\,2},\hat{\sigma}_D^{\,2}\right)=-\tfrac{1}{6n}\sigma_D^{\,4},\quad \mathrm{cov}\left(\hat{\sigma}_C^{\,2},\hat{\sigma}_D^{\,2}\right)=-\tfrac{3}{2n}\sigma_D^{\,4}$$

For Design 2

$$\mathrm{var}\left(\hat{\sigma}_A^{\,2}\right)=\tfrac{2}{n}\left(\tfrac{1}{16}\sigma_D^{\,4}+\tfrac{7}{64}\sigma_C^{\,4}+\tfrac{3}{16}\sigma_C^{\,2}\sigma_D^{\,2}+\tfrac{1}{4}\sigma_B^{\,4}+\tfrac{1}{4}\sigma_B^{\,2}\sigma_D^{\,2}+\tfrac{3}{8}\sigma_B^{\,2}\sigma_C^{\,2}\right)$$
$$+\tfrac{2}{n-1}\left(\tfrac{1}{4}\sigma_D^{\,2}+\tfrac{3}{8}\sigma_C^{\,2}+\tfrac{1}{2}\sigma_B^{\,2}+\sigma_A^{\,2}\right)^2$$
$$\mathrm{var}\left(\hat{\sigma}_B^{\,2}\right)=\tfrac{2}{n}\left(\tfrac{7}{8}\sigma_D^{\,4}+\tfrac{9}{8}\sigma_C^{\,4}+\tfrac{15}{8}\sigma_C^{\,2}\sigma_D^{\,2}+\sigma_B^{\,4}+\sigma_B^{\,2}\sigma_D^{\,2}+\tfrac{3}{2}\sigma_B^{\,2}\sigma_C^{\,2}\right)$$
$$\mathrm{var}\left(\hat{\sigma}_C^{\,2}\right)=\tfrac{2}{n}\left(2\sigma_D^{\,4}+\sigma_C^{\,4}+2\sigma_C^{\,2}\sigma_D^{\,2}\right),\quad \mathrm{var}\left(\hat{\sigma}_D^{\,2}\right)=\tfrac{2}{n}\sigma_D^{\,4}$$
$$\mathrm{cov}\left(\hat{\sigma}_A^{\,2},\hat{\sigma}_B^{\,2}\right)=-\tfrac{2}{n}\left(\tfrac{1}{8}\sigma_D^{\,4}+\tfrac{1}{4}\sigma_C^{\,4}+\tfrac{3}{8}\sigma_C^{\,2}\sigma_D^{\,2}+\tfrac{1}{2}\sigma_B^{\,4}+\tfrac{1}{2}\sigma_B^{\,2}\sigma_D^{\,2}+\tfrac{3}{4}\sigma_B^{\,2}\sigma_C^{\,2}\right)$$
$$\mathrm{cov}\left(\hat{\sigma}_A^{\,2},\hat{\sigma}_C^{\,2}\right)=0,\quad \mathrm{cov}\left(\hat{\sigma}_B^{\,2},\hat{\sigma}_C^{\,2}\right)=-\tfrac{2}{n}\left(\sigma_D^{\,4}+\tfrac{3}{4}\sigma_C^{\,4}+\tfrac{3}{2}\sigma_C^{\,2}\sigma_D^{\,2}\right)$$
$$\mathrm{cov}\left(\hat{\sigma}_A^{\,2},\hat{\sigma}_D^{\,2}\right)=0,\quad \mathrm{cov}\left(\hat{\sigma}_B^{\,2},\hat{\sigma}_D^{\,2}\right)=\tfrac{1}{2n}\sigma_D^{\,4},\quad \mathrm{cov}\left(\hat{\sigma}_C^{\,2},\hat{\sigma}_D^{\,2}\right)=-\tfrac{2}{n}\sigma_D^{\,4}$$