방향성을 갖는 비정렬 삼각형격자를 이용한 단조 유선 Upwind 유한요소해석

The Monotone Streamline Upwind Finite Element Method Using Directionally Aligned Unstructured Grids

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Abstract

Rice's monotone streamline upwind finite element method, which was proposed to treat convection-dominated flows, is applied to the linear triangular element. An alignment technique of unstructured grids with given velocity fields is used to prevent the interpolation error produced in evaluating the convection term in the upwind method. The alignment of grids is accomplished by optimizing a target function defined with the inner-product of a properly chosen side vector in the element with the velocity field. Two pure advection problems are considered to demonstrate the superiorities of the present approach in solving the convection-dominated flow on the unstructured grid. Solutions obtained with aligned grids are much closer to the exact solutions than those with initial regular grids. The capability of the present approach in predicting the appearance of the secondary vortex in the laminar confined jet impingement is shown by comparing streamlines to those produced by SIMPLE on a highly stretched grid toward the impingement plate.

1. Introduction

In the finite element approaches, the Galerkin method is commonly used to discretize the spatial domain. It is, however, well known that the Galerkin method, which uses the shape function as the weighting function, gives unphysical oscillatory solutions when it is applied to convection-dominated flows [1]. Various upwind finite element schemes have been presented to remove such spatial oscillatory behaviors [1, 2, 3]. The mainstream of those work is to include the upwind effect with modifying the Galerkin weighting function. Chee and Kwon have proposed an optimal weighting function based on the variational principles to solve the compressible Euler equations [2]. Hughes et al. have present the streamline upwind Petrov-Galerkin method (SUPG) of which weighting functions are discontinuous across the element boundaries [1]. Of those methods, SUPG has been commonly used either in the compressible or in the incompressible finite element computations and gives reasonable solutions on rectangular elements.

The application of SUPG, however, is not clear to triangular elements. Mizukami applied SUPG scheme to linear triangular elements and modified his scheme to make sure the satisfaction of the discrete maximum principle [3]. The modification is accomplished by adding different constant values to each nodal weighting function depending on the direction of the velocity in the element. The main shortcoming of SUPG on triangular elements is that it is limited to the triangular elements of the weakly acute type, i.e., all inner angles are less than or equal to $\frac{1}{2}\pi$.

Rice presented the monotone streamline upwind method which directly approximated to the advection terms themselves, rather than modifying the weighting function [4]. In the paper, bilinear rectangular elements were used to demonstrate the effect of the proposed method. This scheme is quite simple to implement with existing finite element methods and can be directly applied to the triangular element. It is, therefore, thought to be promising upwind scheme to solve the convection-dominated flows with the unstructured triangular element.

The monotone streamline upwind method is one of the first-order upwind methods. It contains the numerical diffusion of its own. Moreover, it is likely to have additional diffusive behaviors due to the fact that it calculates the convection term using the variables between two points which are away from nodal points of the element. It is thought that it is important, if possible, to make calculations of the convection term take place between nodal points [5]. A nearly exact solution is recovered for the convection term if it is differenced between nodal points in the element. It is the motivation of present work.

Alignment of the computational grid with the vector field associated with flow streamlines has long been known to be effective means of improving the accuracy of calculation [6]. Brackbill used a variational formulation to occur directionally controlled node movement. The alignment of elements can be accomplished by using the optimization, which has been used to get smooth grids or adapted grids in the finite difference field [7]. To achieve our goals, the target function to be optimized is defined as a linear combination of the regularity of elements and the inner-product of the characteristic velocity with a properly chosen side vector in each element.

The purpose of this paper is to introduce an application of Rice's upwind finite element method to the triangular unstructured element and to show improvements of the accuracy in the upwind method by using the alignment technique of grids.

2. The monotone streamline upwind method [4]

Let us consider the scalar transport equation with known velocity fields and constant properties, especially with large **Pe** number.

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = \frac{1}{\mathbf{Pe}}\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right]. \quad (1)$$

The finite element equations are obtained by applying the Galerkin weighted residual method. In the present paper, the flow domain is discretized using two-dimensional linear triangular elements.

The element equations are given as follows:

$$\int W\left[\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y}\right] dA = \int \frac{1}{\mathbf{Pe}}\left[\frac{\partial W}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial W}{\partial y}\frac{\partial \phi}{\partial y}\right] dA. \quad (2)$$

The time integration term and diffusion term appearing in Eq.(2) are treated in the quite standard manner. As in [4], the element advection term is calculated assuming that

$$u_s\frac{\partial \phi}{\partial s} = \text{constant} \quad (3)$$

along a streamline on an element. With this assumption, the element advection term is approximated as:

$$\left[u_s\frac{\partial \phi}{\partial s}\right]\int W dA. \quad (4)$$

Consider the linear triangular element illustrated in Fig.1. The downwind node, I1 in the figure, is defined as the node of which the negative velocity vector points back into the element. If one recalls that the streamline which passes through the downwind node is represented as a straight line in the linear triangular element, the upstream point p, which is located on the opposite side of the downwind node, is algebraically determined.

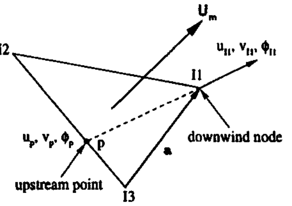


Figure 1. Downwind node and upstream point in a triangular element.

Once two needed points are located, the advection term for the node in Eq.(4) is approximated as:

$$\frac{u_s}{\Delta s}(\phi_{\mathrm{I1}} - \phi_{\mathrm{p}})\int W dA \quad (5)$$

With the help of the shape function, Eq.(5) can be implicitly formulated into the finite element framework.

3. Optimization of a designed target function

Grid generations and solution adaptations using optimization consist in defining a measure of the deformation between the current cell and the reference cell and in minimizing the global target function obtained by addition of all these local contributions [7]. Considering the characteristics of the grid generation using optimization, we think that one can get an aligned set of elements with the vector field if constructing a principle which is well-designed to drive the alignment of grids. In the present work, an inner-product in each element is chosen as a functional to be optimized. Assuming that the side I3-I1 is to be aligned with the centroid velocity $\mathbf{U_m}$ in Fig.1, we define the target function for this element as:

$$\mathbf{ALN}_e = (|\mathbf{U_m}| \cdot |\mathbf{a}|)^2 - (\mathbf{U_m} \cdot \mathbf{a})^2. \quad (6)$$

It is clear that $\mathbf{ALN}_e$ is always greater than zero and if side I3-I1 is exactly aligned with $\mathbf{U_m}$, Eq.(6) becomes equal to zero.

It is, however, not satisfactory only with the target function defined above because it is possible to produce a set of overlapped elements. In this work, therefore, the definition of measure which was proposed by Jacquotte in 1988 is to be added to Eq.(6) in order to remove the overlapping of elements [7]. Jacquotte's functional is expressed as:

$$\mathbf{RG}_e = \alpha(I_1 - 2J) + (1-\alpha)(J-1)^2 \quad (7)$$

where α is a factor to be determined to verify the existence of the minimum of the functional. If there is no special refer, α is set to be 0.3 in this paper.

The global objective function F is obtained by forming a weighted linear combination of local measures, Eq.(6) and Eq.(7), and summing them up over all elements over the domain:

$$F = \sum_{e} [G_1 \mathbf{ALN}_e + G_2 \mathbf{RG}_e], \qquad (8)$$

where G_1 and G_2 are the scalar weight parameters enabling a trade-off between the grid regularity and the alignment of elements.

The global objective function is then rewritten as a function of the vector V containing the physical coordinates of all the grid points:

$$V = \{(x_i, y_i) : 1 \leq i \leq np\} \qquad (9)$$

where np is the number of points in the domain. Indeed, Eq.(8) is a fourth-order polynomial of coordinates of all these node points. Unconstrained minimization of the function F of $2 \times np$ variables is performed using the Fletcher-Reeves conjugate gradient method. The minimization is a succession of one-dimensional minimization problems. The numerical procedure of the present approach consists in evaluating the gradient from the previously solved solutions on the initial grid and in determining the weight parameter G_1 as a function of coordinates of nodal points. Finally, with the help of optimization, the elements are partly aligned with the vector field based on the given distribution of the weight parameter.

4. Numerical results

In all cases, triangular elements are generated in the unstructured manner using the advancing front techniques.

4.1 Convection skew to the mesh [3, 4]

The problem is one of the pure advection transport cases, i.e., the Peclet number is infinite. The problem statement and boundary conditions are shown in Fig.2. The flow is unidirectional ($\theta = \pi/3$) and constant ($|\vec{u}| = 1$). The Peclet number is taken to be 10^7. An employed initial mesh is shown in Fig.3 (a). Contours of the transported variable obtained with the initial mesh are shown in Fig.3 (b). The solution is smeared along streamlines due to the numerical diffusion. In Fig.3 (c), a partly aligned grid with the direction of the velocity is shown. Elements in the region where the solution has the steep gradient are aligned with streamlines. As can be seen in Fig.3 (d), the solution solved with the aligned grid becomes steeper than that obtained with the initial regular grid.

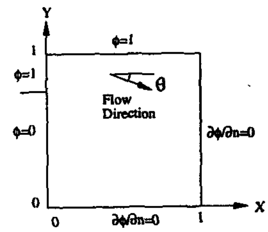
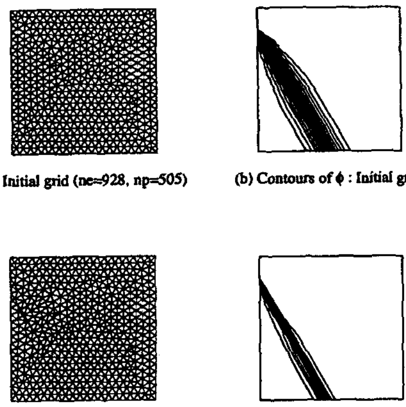


Figure 2. Convection skew to the mesh



(a) Initial grid (ne=928, np=505)

(b) Contours of φ : Initial grid

(c) Partly aligned grid with the direction of velocities

(d) Contours of φ : Aligned grid

Figure 3. Convection skew to the mesh

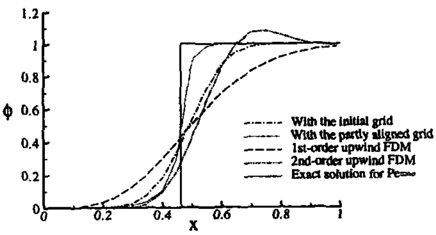
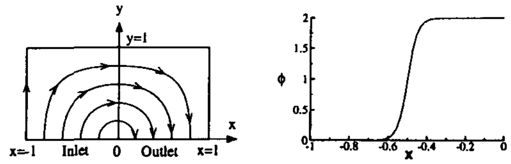


Figure 4. Comparisons of ϕ profiles at $y = 0$.



(a) Geometry and streamlines (b) Inlet distribution of ϕ

Figure 5. Smith and Hutton test case

Fig.4 is a plot of ϕ profiles at $y = 0$. The exact solution for $\mathbf{Pe} = \infty$ and results of finite difference methods on a 21×21 regular grid are drawn together for comparison. It is seen that the solution of the present approach with the initial grid has nearly same order of accuracy as that obtained by the finite difference method with the second-order upwind. Seeing that the second-order finite difference method is likely to have wiggles near the sharp gradient without a limiter, it is thought that Rice's upwind method gives a reasonable solution on the triangular unstructured grid. The improvement of the accuracy with the alignment is clearly seen in the figure.

4.2 Smith and Hutton test case [4]

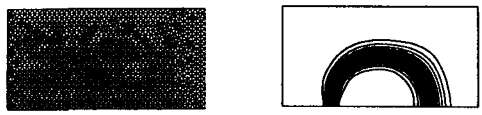
The flow domain is shown in Fig.5 (a). The Peclet number is taken to be infinite. The velocity field is given. The boundary condition of ϕ along the inlet is given by

$$\phi(x, y = 0) = 1 + \tanh\left[10(2x + 1)\right]. \qquad (10)$$

and shown in Fig.5 (b). Inspite of the strongly curving streamlines, the inlet profile should be transported along the streamlines without any diffusion for the infinite Peclet number and it should have the mirror-imaged profile of the inlet in the outlet. Fig.6 (a) is a initial grid which is regularly generated and Fig.6 (b) shows its contours of ϕ. Note that the initial profile is diffused along the streamlines. Fig.7 (a) is a partly aligned grid. Fig.7 (b) shows that the current method gives a considerable improvement in removing the diffusive behavior in the exit. Also, note that the current method does not exhibit any spatial oscillations. To show the improvement more clearly, the outlet profile of ϕ is compared to the exact solution and the initial solution in Fig.8. It shows that although there exist smearings in the solutions yet, the profile obtained with the alignment of elements is closer to the exact profile than that obtained without the alignment.

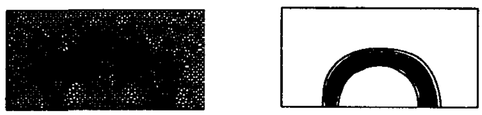
4.3 Laminar jet impingement flow

Jet impingment flows are frequently used in industrial practice for their excellent heat and mass



(a) Initial grid (ne=1854, np=988) (b) Contours of ϕ with the initial grid

Figure 6. Smith and Hutton test case:Initial unstructured grid



(a) Partly aligned grid (ne=1854, np=988) (b) Contours of ϕ with the partly aligned grid

Figure 7. Smith and Hutton test case:Partly aligned grid

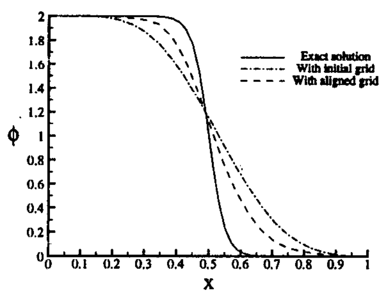


Figure 8. Comparisons of outlet profiles for infinite Peclet number.

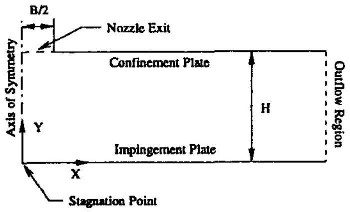
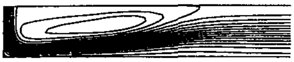


Figure 9. Coordinate system and boundaries of the jet system.



(a) Streamlines: 62×21 Grids

(b) Streamlines: 62×35 Grids

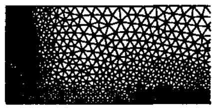
Figure 10. Jet impingements using SIMPLE method

transfer characteristics, where localized and controlled surface transfer is desirable. In the practical aspect, it is important to predict the Reynolds number with which the secondary vortex appears just above the impingment plate. It is a reason that if there appears a secondary vortex on the bottom plate which is usually to be cooled by the impinging jet, it affects abruptly its heat transfer behavior.

The system of jet impingement considered in this study is shown in Fig.9. The jet issues from a slot tube of width B with an average velocity of $\mathbf{U}$. The confinement plate is located parallel to and at a distance H from the impingement plate. The lengths of the confinement and the impingement plates are set to the normalized length of 40 in this study. The parabolic velocity profile with an average velocity of $\mathbf{U}$ is given in the nozzle exit as a boundary condition. The outlet boundary is located far enough downstream for conditions to be substantially developed. No slip conditions are imposed on the plates and the conventional symmetric condition is given on the plane over the slot-jet axis. The two-dimensional laminar incompressible Navier-Stokes flow is solved using a segregated finite element method [8]. In the computation, the convection term is discretized by the monotone streamline upwind scheme [4]. We only consider the case with H of 2 and $\mathbf{Re}$ of 200 in this study since our purpose is only to show the improvement of solutions by the alignment of elements.

At first, in order to compare the finite element results, we get the streamline contours with same problem parameters using SIMPLE finite volume method [9]. The hybrid scheme is adopted as an upwind method, which switches the central difference to the first-order upwind difference where the convection term becomes large. Two different grid systems are used to show that if one uses more grid points in the direction of Y in this case, he predicts the appearance of the secondary vortex just above the impingement plate. Fig.10 shows the streamlines with the 62×21 and

(a) Unstructured elements



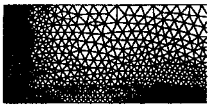
(b) Close view of the stagnation region

(c) Streamlines

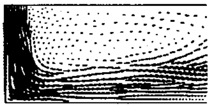
Figure 11. Jet impingements using Rice's upwind method with the initial grid

62×35 grids. As can be seen in the figure, the secondary vortex appears above the impingement plate when grid points are added in Y-direction. The unstructured element system in the front of the domain is shown in Fig.11 (a), (b). Elements are gathered just above the impingement plate to capture the secondary vortex if there exists. In Fig.11 (c), the resulting streamlines are shown. It is not possible to get the secondary vortex only using the monotone streamline upwind method with a unstructured grid. The aligned unstructured elements obtained by the present alignment method are shown in Fig.12 (a), (b). Fig.12 (c) is a velocity vector plot in the jet-issuing region. It is seen that elements are aligned with the direction of velocities. The resulting streamlines are shown in Fig.12 (d). The secondary vortex ap-

(a) Aligned elements



(b) Close view of the stagnation region



(c) Velocity vectors

(d) Streamlines

Figure 12. Jet impingements using Rice's upwind method with the aligned grid

pears on the plate. In the qualitative aspect, it is apparent that the present method gives similar solutions as those obtained by the finite volume method with a highly stretched grid.

5. Conclusion

The monotone streamline upwind finite element method, which was originally used on the bilinear rectangular element in [4], is applied to the linear triangular element. It gives non-oscillatory solutions in solving the flows with high Peclet number with the triangular element. It, however, has additional diffusive behaviors produced in evaluating the convection term itself. It is expected that if elements are aligned with the velocity field, the calculation of the convection term is closer to be exact. An alignment method of the unstructured grid using optimization is, therefore, presented.

The target function is defined as a linear combination of the regularity of elements and the inner-product of the characteristic velocity with a properly chosen side vector in the element. It is optimized using the Fletcher-Reeves conjugate gradient method.

Numerical results show the potential of the new approach in terms of the exactness of the solution in the pure advection cases. After the alignment of the initial grids, solution contours become much closer to the exact solutions. In particular, the alignment enables the monotone streamline upwind scheme to predict the appearance of the secondary vortex in the laminar confined jet impingement flow, which is likely to be smeared in the regular grid due to the numerical diffusion, without adding or adaptive remeshing of elements.

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