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일정 흐름장에서의 파랑과 다공질 탄성 해저지반의 상호작용
Interactions of Wave and Poro-elastic Seabed under Uniform Current

김 범 형* 이 길 성** 박 우 선***

1. Introduction

Ocean seabed is usually covered with various types of marine soils. A marine soil is a mixture of two phases: soil particles that forms an interlocking skeletal frame, pore fluids that occupy a major portion of pore space.

When gravity water waves propagate over a porous movable seabed, a hydrodynamic pressure on the fluid-seabed interface and fluid flow in the porous medium are induced. Then the pore fluid transmits a force to the skeletal frame in the form of effective stresses, and wave-induced effective stresses cause deformation of the porous medium. The hydrodynamic pressure at the surface of the porous seabed induces elastic waves - compressional waves and shear wave - in the porous seabed, which physically mean a flow within the seabed and a dynamic deformation of soil skeletal frame. During this process, water wave energy may be dissipated by several damping mechanisms such as bottom friction, viscous friction of fluid (percolation), Coulomb friction between soil particles (Yamamoto, 1983). The wave length of water waves is also modified by the seabed response resulted from wave actions. Thus, the problem of wave-seabed interaction may be considered as a problem of forced vibration by water waves of a poro-elastic bed. Therefore, the boundary value problem should be treated as the coupled problem of water waves and elastic waves in the seabed.

The linear wave theory is considered to model wave motions in the fluid region, and the behaviors of skeletal frame of soil and pore fluid in the seabed are based on the linear theory of elastic wave propagation in porous medium by Biot (1962). Yamamoto et al. (1978) and Madsen (1978) obtained analytically the behaviors of the unbounded porous medium by water waves using Biot's theory (1962).

In the present study, an implicit closed form solution for the interaction problem of water wave and poro-elastic seabed under uniform current has been obtained. Without current effects, the derived solution reduces to the analytic solution of Yamamoto (1983). The influences on wave-seabed interaction according to the seabed conditions and current variations are also investigated.

2. Gravity water waves on the porous seabed

A regular wave train of amplitude, $H/2$, and angular frequency, ω, propagating on the porous seabed of constant water depth, h, under uniform current is considered to formulate the interaction problem of waves and the seabed as shown on Fig. 1. A Cartesian coordinate system (x, z) with x measured in the direction of wave propagation, and z vertically upwards from the stillwater level is adopted.

Assuming time-harmonic motion of fluid particles, the free surface fluctuation, η, can be expressed as

* 박사과정수료, 서울대학교 토목공학과(금호엔지니어링(주) 항만부)
** 교수, 서울대학교 토목공학과
*** 선임연구원, 한국해양연구소 연안공학연구부

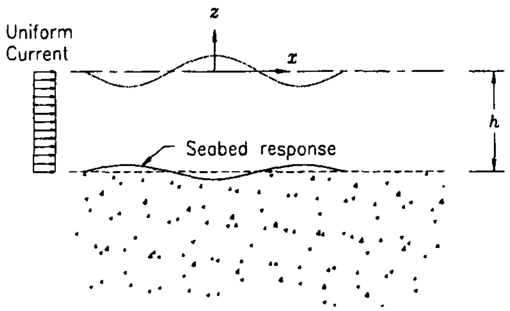


Fig. 1 Definition sketch

$$\eta = \frac{H}{2} e^{i(kx - \omega t)} \tag{1}$$

where, k is complex-valued wave number and $i = \sqrt{-1}$.

The fluid is assumed to be incompressible, and the flow is irrotational. The fluid motion, therefore, can be described by a velocity potential, $\Phi(x, z; t)$, which satisfies the Laplace equation

$$\nabla^2 \Phi(x, z; t) = 0 \tag{2}$$

Using a small amplitude wave theory, the dynamic and kinematic free surface boundary conditions under uniform current, U_o, may be linearized as (Dean and Darlymple, 1984)

$$\eta = -\frac{1}{g}\left(\frac{\partial \Phi}{\partial t} + U_o \frac{\partial \Phi}{\partial x}\right) \quad \text{at } z = 0 \tag{3}$$

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \eta}{\partial t} + U_o \frac{\partial \eta}{\partial x} \quad \text{at } z = 0 \tag{4}$$

in which g = gravitational acceleration.

At the bed surface, the vertical velocity of the fluid particle should be equal to that of the pore water.

$$\frac{\partial \Phi}{\partial z} = \frac{\partial u_z}{\partial t} + \frac{\partial w_z}{\partial t} \quad \text{at } z = -h \tag{5}$$

in which, u_z = vertical displacement of the soil at the seabed; w_z = relative vertical pore fluid displacement.

Using separation of variables, the general solution of velocity potential, Φ, which satisfies the Laplace equation (2) and the boundary conditions (3) and (4), can be obtained as

$$\Phi = -i\frac{H}{2}\left(\frac{g}{\omega\left(1 - \frac{U_o k}{\omega}\right)} \cosh kz + \frac{\omega}{k}\left(1 - \frac{U_o k}{\omega}\right) \sinh kz\right) e^{i(kx - \omega t)} \tag{6}$$

where, the complex-valued wave number, k, which is modified by the seabed response, is to be determined from the following dispersion relation obtained from the boundary condition (5).

$$\omega^2 = (gk \tanh kh) \cdot \left[\frac{1}{\left(1 - \frac{U_o k}{\omega}\right)^2}\right] \cdot \left[\frac{1}{1 - \frac{U_z + W_z}{H/2}}\right] \tag{7}$$

in which, U_z = complex-valued amplitude for vertical displacement of soil skeletal frame; W_z = complex-valued amplitude for relative vertical displacement of pore fluid.

3. Seabed response to water waves

Biot (1962) presented the linear theory on the elastic waves propagating in fluid-filled poro-elastic media. Combining the stress-strain relations with Darcy's law and stress equilibrium equation, the equations of motion for soil skeletal frame and pore fluid are governed by

$$G\nabla^2 \boldsymbol{u} + (H-G)\nabla\varepsilon + C\nabla\zeta = \frac{\partial^2}{\partial t^2}(\rho\boldsymbol{u} + \rho_f \boldsymbol{w}) \tag{8}$$

$$\nabla(C\varepsilon + M\zeta) = \frac{\partial^2}{\partial t^2}(\rho_f \boldsymbol{u} + m\boldsymbol{w}) + \frac{\eta_f}{k_s}\frac{\partial \boldsymbol{w}}{\partial t} \tag{9}$$

in which H, M, C = Biot's elastic modulus; $\boldsymbol{u}$ = displacement vector of soil skeletal frame of which components are u_x and u_z; $\boldsymbol{w}$ = relative displacement vector of pore fluid of which components are w_x and w_z; ε = strain of soil skeletal frame; ζ = volumetric strain of pore fluid; G = shear modulus of the soil; ρ = total density of the soil; ρ_f = density of pore fluid; m = virtual mass of skeletal frame in accelerated flow field; η_f = viscosity of the pore fluid (kg/m · s); and $k_s = \chi\eta_f/\rho_f g$; χ = permeability of the soil (m/sec)

There are two solutions for Biot's equations (8) and (9), which are for motion of an fluid-filled homogeneous, isotropic, and elastic medium (Prakash et al., 1981). One solution describes the propagation of an irrotational wave (compressional wave, dilatational wave), while the other describes the propagation of a wave of pure rotation (shear wave, equivoluminal wave).

Introducing displacement potentials ϕ_c^u and ϕ_c^w for compressional waves, and ϕ_s^u and ϕ_s^w for shear wave, the displacement of soil skeletal frame, $\boldsymbol{u}$, and the relative displacement of pore fluid, $\boldsymbol{w}$, in the x- and z-directions, respectively, can be written in terms of four potential functions (Yamamoto, 1983).

$$\boldsymbol{u} = \nabla\phi_c^u - \triangle\phi_s^u = \left(\frac{\partial\phi_c^u}{\partial x} - \frac{\partial\phi_s^u}{\partial z}\right)\vec{i} + \left(\frac{\partial\phi_c^u}{\partial z} + \frac{\partial\phi_s^u}{\partial x}\right)\vec{k} \tag{10}$$

$$\boldsymbol{w} = \nabla\phi_c^w - \triangle\phi_s^w = \left(\frac{\partial\phi_c^w}{\partial x} - \frac{\partial\phi_s^w}{\partial z}\right)\vec{i} + \left(\frac{\partial\phi_c^w}{\partial z} + \frac{\partial\phi_s^w}{\partial x}\right)\vec{k} \tag{11}$$

in which $\nabla(\cdot) = \partial(\cdot)/\partial x\,\vec{i} + \partial(\cdot)/\partial z\,\vec{k}$ and $\triangle(\cdot) = \partial(\cdot)/\partial z\,\vec{i} - \partial(\cdot)/\partial x\,\vec{k}$.

Since waves motion is harmonic as given in Eq.(1), the displacement potentials, ϕ_c^u, ϕ_c^w, ϕ_s^u and ϕ_s^w are assumed to be harmonic in both time and x-direction.

$$\phi_c^u = \Psi_c^u(z)\, e^{i(kx-\omega t)} \tag{12}$$

$$\phi_c^w = \Psi_c^w(z)\, e^{i(kx-\omega t)} \tag{13}$$

$$\phi_s^u = \Psi_s^u(z)\, e^{i(kx-\omega t)} \tag{14}$$

$$\phi_s^w = \Psi_s^w(z)\, e^{i(kx-\omega t)} \tag{15}$$

Substitution of the potential functions Eqs. (12) and (13) for compressional waves and Eqs. (14) and (15) for shear wave into the governing Eqs. (8) and (9) lead to the ordinary differential equations for $\Psi_c^u(z)$, $\Psi_c^w(z)$, $\Psi_s^u(z)$ and $\Psi_s^w(z)$.

$$\begin{bmatrix} H(k^2-D^2) - \rho\omega^2 & C(k^2-D^2) - \rho_f\omega^2 \\ C(k^2-D^2) - \rho_f\omega^2 & M(k^2-D^2) - m\omega^2 - i\frac{\eta_f}{k_s}\omega \end{bmatrix} \begin{pmatrix} \Psi_c^u(z) \\ \Psi_c^w(z) \end{pmatrix} = 0 \tag{16}$$

$$\begin{bmatrix} -G(k^2-D^2) + \rho\omega^2 & \rho_f\omega^2 \\ \rho_f\omega^2 & m\omega^2 + i\frac{\eta_f}{k_s}\omega \end{bmatrix} \begin{pmatrix} \Psi_s^u(z) \\ \Psi_s^w(z) \end{pmatrix} = 0 \tag{17}$$

We have obtained the general solutions: Φ given by equation (6) for the fluid motion on the porous seabed ($-h < z < 0$); ϕ_c^u, ϕ_c^w given by Eqs. (24), (25) for the motion of the seabed ($z < -h$) associated with the compressional waves; and ϕ_s^u , ϕ_s^w given by Eqs. (26), (27) for the motion of the seabed ($z < -h$) associated with the shear wave.

The boundary conditions for the seabed at the bed surface are that the vertical effective stress, σ_z', is zero, that the shear stress, τ_{xz}, is zero, and that the wave induced hydrodynamic pressure, p ($p > 0$ for compression), is transmitted continuously from the sea to the pores in the seabed;

$$\sigma_z' = \sigma_z + p = 0 \qquad \text{at } z = -h \tag{31}$$

$$\tau_{xz} = 0 \qquad \text{at } z = -h \tag{32}$$

$$p_{(in\ bed)} = -\rho_f \frac{\partial \Phi}{\partial t} \qquad \text{at } z = -h \tag{33}$$

The general solutions obtained contain four unknown complex constants k, a_1, a_2, a_3, which can be determined from the boundary conditions (5) and (31) ~ (32) at the sea-seabed interface. Using four boundary conditions, the four simultaneous equations are constituted as

$$\begin{bmatrix} \{s_1^2[c_1(M-C)+(C-H)]+2Gk^2\} & \{s_2^2[c_2(M-C)+(C-H)]+2Gk^2\} & 2iGk\lambda_3 \\ 2iGk\lambda_1 & 2iGk\lambda_2 & -G(2k^2-s_3^2) \\ (c_1M+C)s_1^2 & (c_2M+C)s_2^2 & 0 \\ \lambda_1(1+c_1) & \lambda_2(1+c_2) & ik(1+c_3) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \rho_f \frac{H}{2} g\left(\frac{1}{(1-\frac{U_o k}{\omega})}\cosh kh - \frac{\omega^2}{kg}\left(1-\frac{U_o k}{\omega}\right)\sinh kh \right) \\ \frac{H}{2}\left(\left(1-\frac{U_o k}{\omega}\right)\cosh kh - \frac{gk}{\omega^2(1-\frac{U_o k}{\omega})}\sinh kh \right) \end{bmatrix} \tag{34}$$

Four unknown complex constants k, a_1, a_2, a_3 can be determined from Eq. (34) by iteration method such as Newton's method. The components of total stress and pore pressure can be obtained as

$$\begin{bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \\ p \end{bmatrix} = \begin{bmatrix} -\{(H+c_1C)s_1^2+2G\lambda_1^2\} & -\{(H+c_2C)s_2^2+2G\lambda_2^2\} & -2iGk\lambda_3 \\ -\{(H+c_1C)s_1^2-2Gk^2\} & -\{(H+c_2C)s_2^2-2Gk^2\} & 2iGk\lambda_3 \\ 2iGk\lambda_1 & 2iGk\lambda_2 & -G(2k^2-s_3^2) \\ (c_1M+C)s_1^2 & (c_2M+C)s_2^2 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} a_1 e^{\lambda_1(z+h)} \\ a_2 e^{\lambda_2(z+h)} \\ a_3 e^{\lambda_3(z+h)} \end{bmatrix} e^{i(kx-\omega t)} \tag{35}$$

in which $s_n^2 = k^2 - \lambda_n^2$ ($n = 1, 2, 3$).

in which, D is the differential operator, $D(\cdot) = d(\cdot)/dz$.
From this ordinary differential equations, four characteristics roots, $\pm\lambda_1$ and $\pm\lambda_2$, for compressional waves and, two characteristics roots, $\pm\lambda_3$, for shear wave, which are representing vertical decaying parameter of elastic waves, can be obtained, i.e.

$$\lambda_n = k(1 - \xi_n^2)^{1/2} \qquad n = 1, 2, 3 \tag{18}$$

The parameter ξ_n are the Mach numbers which are the ratios of the phase velocity, ω/k, of the gravity water waves and the propagation velocities, V_n, of the three kinds of elastic waves in an unbounded porous medium;

$$\xi_n = \frac{\omega/k}{V_n} \qquad n = 1, 2, 3 \tag{19}$$

The propagation speeds of the two compressional waves in unbounded media are given by

$$V_1 = \left(\frac{2(HM - C^2)}{(\rho M + \overline{m}H - 2\rho_f C) - [(\overline{m}H - \rho M)^2 + 4\rho_f(\rho_f M - \overline{m}C)H + 4\rho(\overline{m}C - \rho_f M)C]^{1/2}}\right)^{1/2} \tag{20}$$

$$V_2 = \left(\frac{2(HM - C^2)}{(\rho M + \overline{m}H - 2\rho_f C) + [(\overline{m}H - \rho M)^2 + 4\rho_f(\rho_f M - \overline{m}C)H + 4\rho(\overline{m}C - \rho_f M)C]^{1/2}}\right)^{1/2} \tag{21}$$

and the propagation velocity of the shear wave is given as

$$V_3 = \left(\frac{G}{\rho}\right)^{1/2}\left(1 - \frac{\rho_f^2}{\rho\,\overline{m}}\right)^{-1/2} \tag{22}$$

in which

$$\overline{m} = m + i\frac{\eta_f}{k_s\omega} \tag{23}$$

Since elastic waves propagate downward and decay from the sea-seabed interface in a semi-infinite half space, the general solutions of the displacement potentials ϕ_c^u and ϕ_c^w for the compressional waves and ϕ_s^u and ϕ_s^w for shear wave in the seabed can be expressed as by the linear summation of the fundamental solutions $\exp[\lambda_1 z]$, $\exp[\lambda_2 z]$ and $\exp[\lambda_3 z]$.

$$\phi_c^u = \left[a_1 e^{\lambda_1(z+h)} + a_2 e^{\lambda_2(z+h)}\right] e^{i(kx-\omega t)} \tag{24}$$

$$\phi_c^w = \left[b_1 e^{\lambda_1(z+h)} + b_2 e^{\lambda_2(z+h)}\right] e^{i(kx-\omega t)} \tag{25}$$

$$\phi_s^u = a_3 e^{\lambda_3(z+h)} e^{i(kx-\omega t)} \tag{26}$$

$$\phi_s^w = b_3 e^{\lambda_3(z+h)} e^{i(kx-\omega t)} \tag{27}$$

The coefficients a_n and b_n are not independent. The dependence can be determined by substitution of Eqs. (24)~(27) into the governing Eqs. (8) and (9).

$$b_n = c_n \cdot a_n \qquad n = 1, 2, 3 \tag{28}$$

in which c_n are given as

$$c_n = -\frac{\rho}{\rho_f}\frac{\left(V_n^2 - \dfrac{H}{\rho}\right)}{\left(V_n^2 - \dfrac{C}{\rho_f}\right)} \qquad n = 1, 2 \tag{29}$$

$$c_3 = -\frac{\rho_f}{m} \tag{30}$$

4. Interaction analysis and Discussion

To investigate the effect of wave-seabed interaction using the analytical solution derived in this present study, the analyses for the case that water waves with $T = 15$ s and $H^0 = 5$ m propagate on the unbounded sand bed and soft clay bed of water depth $h = 50$ m, $h = 5$ m are carried out, respectively.

Pore pressure at the sea bed and the wave number modified by the seabed response according to the seabed conditions and current variations are summarized in Table 1. It can be concluded that the wave length, L, of water wave over the soft clay bed is shortened compared to the wave length, L_o, of water wave over a rigid bed. The bottom pressure, p_B at the surface of the soft clay bed is larger than that of the sand bed. Especially, the influence of seabed response is increased when the water wave and uniform current have the same progressive direction. This phenomenon was more remarkable for intermediate depth condition (50 m) than shallow water depth condition(5 m).

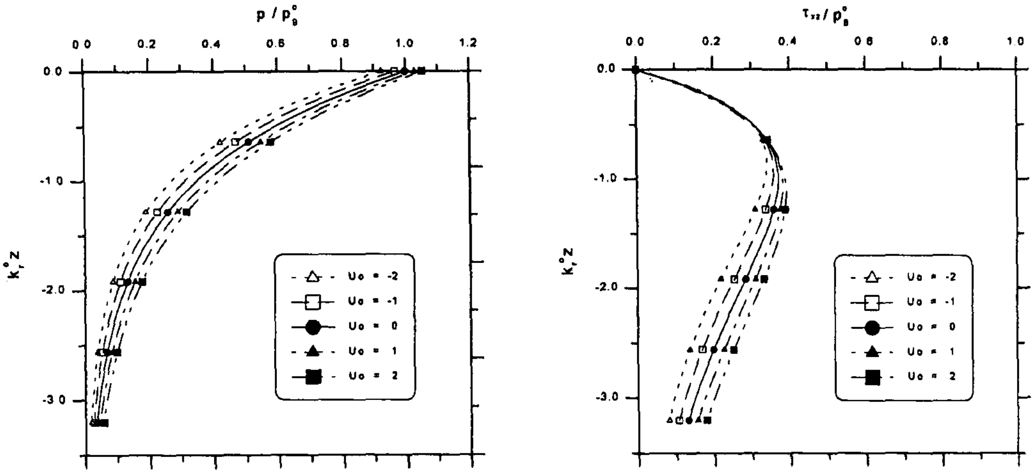
The vertical distributions of max. pore pressure and shear stresses are shown on Fig. 2~5. For intermediate water depth condition, the seabed responses are sensitive to the seabed condition. However, the seabed responses are not sensitive to the seabed condition for shallow water depth. It might be caused that the vertical movement of water particles in shallow water condition is relatively smaller than that of water particles in intermediate water depth.

Table 1 Variation of wave numbers and maximum pore water pressures at seabed w.r.t different current conditions

h (m)	U (m)	H/H^0	Rigid bed		Sand bed		Clay bed		Remarks
			p_B/w_0H^0	k_r/k_r^0	p_B/w_0H^0	k_r/k_r^0	p_B/w_0H^0	k_r/k_r^0	
50	-2	1.228	0.281	1.179	0.282	1.179	0.312	1.236	T = 15 s
	-1	1.098	0.294	1.079	0.294	1.080	0.336	1.153	H^0= 5 m
	0	1.000	0.304	1.000	0.305	1.001	0.359	1.092	
	1	0.923	0.313	0.935	0.313	0.936	0.383	1.046	k_r^0=0.022245 m^{-1}
	2	0.859	0.320	0.879	0.321	0.881	0.409	1.012	
5	-1	1.186	0.487	1.177	0.487	1.178	0.487	1.210	T =15 s
	0	1.000	0.490	1.000	0.490	1.001	0.492	1.039	H^0= 5 m
	1	0.868	0.492	0.871	0.492	0.872	0.498	0.916	
	2	0.768	0.493	0.772	0.494	0.773	0.506	0.823	k_r^0=0.060747 m^{-1}

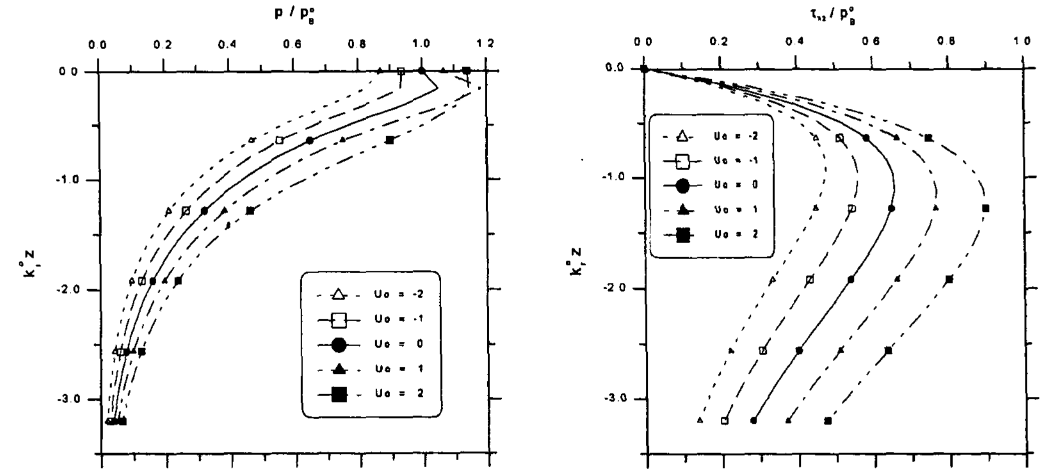
5. Conclusions

In this study, an analytical solution for interactions of poroelastic seabed to water waves under uniform current has been obtained. The wave induced seabed responses according to the seabed condition and current variations are investigated using analytical solution developed in the present study. The responses in the clay bed are large considerably and transferred into the seabed deeply compared to those of the sand bed. The wavelength of the water waves over the soft clay bed is considerably shorter and the pore pressure at the surface of seabed is greater than those of the water wave over the sand bed. For following current conditions, larger pore pressure is induced and the seabed response is more amplified as the current speed increases. This phenomenon was more remarkable for intermediate water depth condition than shallow water depth.



(a) Max. pore pressure (b) Max. shear stress

Fig. 2 Distributions of wave-induced pore water pressures and shear stresses in sand bed under intermediate water depth condition($h = 50\,\mathrm{m}$)



(a) Max. pore pressure (b) Max. shear stress

Fig. 3 Distributions of wave-induced pore water pressures and shear stresses in clay bed under intermediate water depth condition($h = 50\,\mathrm{m}$)

6. References

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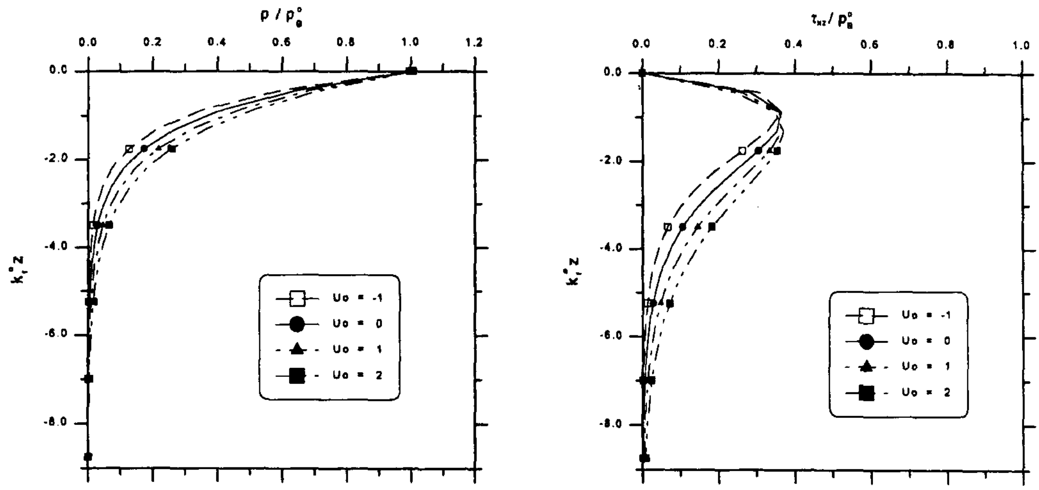
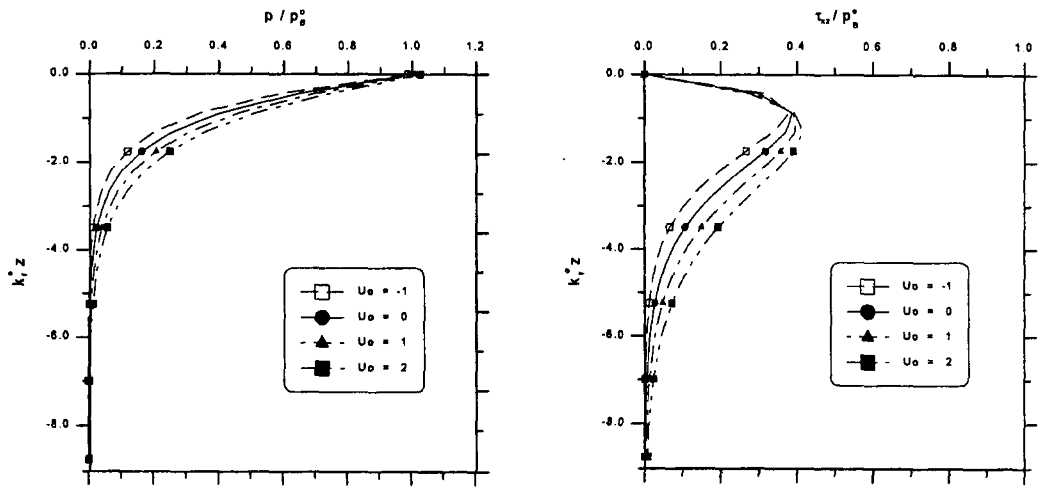


(a) Max. pore pressure (b) Max. shear stress

Fig. 4 Distributions of wave-induced pore water pressures and shear stresses in sand bed under shallow water depth condition($h = 5\,\mathrm{m}$)



(a) Max. pore pressure (b) Max. shear stress

Fig. 5 Distributions of wave-induced pore water pressures and shear stresses in clay bed under shallow water depth condition($h = 5\,\mathrm{m}$)

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