Effect of Transverse Shear Deformation in Thin Elastic Ice Plates

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ABSTRACT

The elastic deflection of thin ice sheets due to bending and shear deformation is considered. The in-plane Young's modulus and the transverse shear modulus are calculated by least squres fit of transverse plate deflection data. Results show that thin ice plates behave predominantly in shear. Previously, the Young's moduli were calculated based on bending theory alone. The Young's moduli of thin model ice sheets, estimated using the bending and shear theory, are more than an order of magnitude greater than calculated previously, and hence are more realistic. Further, the previous ambiguity in the Young's modulus, arising from fitting the data at various distances from the point of loading, is removed by considering shear and bending deformation.

1. INTRODUCTION

Ice plates loaded by static loads but for short duration exhibit elastic behavior. Previously, ice plates were assumed to deform either in bending or in shear only. Both assumptions are based on ice plate deflection observations[1][2]. Based on these data, Kerr[3] suggested that ice plates close to their melting temperature exhibit shear behavior, while colder plates deform predominantly in bending. These two behaviors result in different governing differential equations and contain different elastic moduli.

An *in situ* method for determining the average elastic moduli of a floating ice plate first requires the measurement of deflections due to a prescribed transverse load. The average elastic moduli are then calculated by fitting the load-deflection data with analytical deflection solutions of an elastic plate resting on an elastic foundation. If the plate is assumed to deform in bending, then an average in-plane Young's modulus is calculated. An average transverse shear modulus is obtained if the solution to the shear equation is used. The Young's modulus[1][4][5][6] and the transverse modulus[7][8] of floating ice plates have been calculated in this way.

Timco[4] expressed two important concerns, commenting on the Young's modulus calculated for thin floating model urea ice sheets using the method described above.

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Namely: (i) the Young's modulus can be more than an order of magnitude smaller than $1.3GPa$ - a lower limit for the dynamic Young's modulus he postulates; and (ii) the Young's modulus from the deflection data far away from the loading is higher than the modulus obtained by fitting deflections at the loading. Recently, Elvin[9] showed that theoretically the homogenized transverse shear modulus can increase significantly not only with temperature, but with the thickness of the ice plate as well. The increase in the shear modulus is believed to be due to two factors: (i) grain boundary deformation, and (ii) grain coarsening typically observed through the thickness of the plate. The variation in the shear modulus implies that thin ice plates are more susceptible to shear deformation than thick plates since they have only a few grains through their thickness.

The aim of this paper is to account for both bending and shear deformation effects when calculating the elastic moduli of ice plates. Attention is restricted to thin ice plates. The organization of this paper is as follows. The governing differential equations for calculating the elastic moduli of floating bending plates and floating shear plates are presented in Section 2. The solution of a point load is also given in this section. The governing differential equation and the solution of a floating plate deforming due to bending and shear is presented in Section 3. The method of obtaining both the Young's modulus and transverse shear modulus using least squres fit is described in Section 4. The two elastic moduli are then obtained for a simply supported ice plate in Section 5.

2. DEFORMATION OF A FLOATING ICE PLATE

A floating ice plate responding to short duration transverse loads has been modeled as an elastic plate resting on an elastic foundation, as shown in Figure 1. The elastic foundation models buoyancy, provided the deflections are small. Two independent elastic plate behaviors have been considered; namely the plate was assumed to deform either in bending or in shear. This section presents the governing differential equations associated with these two assumed behaviors and discusses the resulting solutions.

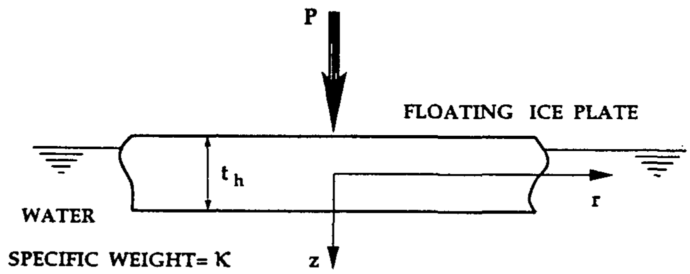


Figure 1: Schematic of a floating ice plate

2.1 Deformation due to bending stresses

The deflected shape of a loaded ice sheet was measured[1]. This data follows the deflection curve predicted by the theoretical Kirchhoff bending plate assumption. Since then the classical Kirchhoff bending plate theory has been applied extensively to the deformation of floating ice sheets of various thickness[3][5][6]. For an isotropic material the governing differential equation[10] of a bending plate resting on an elastic foundation is

$$D\nabla^4 w + \varkappa w = q \tag{1}$$

where ω is the transverse (or vertical) deflection. ∇^4 is the biharmonic operator, q is the transverse distributed load, and $\varkappa$ is the foundation stiffness. For the ice plate the foundation stiffness is due to buoyancy and hence $\varkappa$ is equal to the specific weight of water. The flexural rigidity, D, is given by

$$D = \frac{Et_h^3}{12(1-\nu^2)} \tag{2}$$

where E is the average in-plane Young's modulus, t_h is the ice plate thickness, and ν is the in-plane Poisson ratio.

Next consider the example of a floating bending plate subject to a point load, P at r=0. The boundary conditions on Eq. (1) are

$$\begin{aligned} w &= \frac{dw}{dr} = M = Q = 0 \qquad && at \qquad r = \infty \\ Q &= \lim_{r\to 0}\left(\frac{P}{2\pi r}\right),\ \frac{dw}{dr} = 0 && at \qquad r = 0 \end{aligned} \tag{3}$$

where Q is the shear force/unit length and M is the bending moment/unit length. The solution to Eq. (1) with the boundary conditions in Eq. (3) is given by[10]

$$w(r) = \frac{-P\lambda^2}{2\pi D}\ Kei_o\left(\frac{r}{\lambda}\right) \tag{4}$$

where $Kei_0(x) = Im(K_0(xe^{i\,\pi/4}))$ and K_o is the modified Bessel function of the second kind of order 0. The characteristic length λ is defined as

$$\lambda = \sqrt[4]{\frac{D}{\varkappa}} \tag{5}$$

2.2. Deformation due to shear stresses

The assumtion that ice plates deflect predominantly in shear with negligible bending deformation implies that the transverse shear strain γ_{rz} is

$$\gamma_{rz} = \frac{dw}{dr} \tag{6}$$

The equilibrium condition for the shear plate is

$$\frac{d}{dr}(Qr) = -\varkappa wr + qr \tag{7}$$

where Q is the shear force per unit circumferential length and is defined as

$$Q = \int_0^h \tau_{rz}\, dz \tag{8}$$

where τ_{rz} is the transverse shear stress. If the elastic constitutive law $\tau_{rz} = G\gamma_{rz}$, where G is the average transverse shear modulus of the plate, is used in Eq. (8), with the assumption of Eq. (6), and the result is substituted into Eq. (7), then the governing differential equation for a shear plate on an elastic foundation is obtained as

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \frac{\chi w}{Gt_h} = \frac{q}{Gt_h} \qquad \text{or} \qquad Gt_h \nabla^2 w + \chi w = q \tag{9}$$

The example of a point load, P at $r = 0$ acting on a shear plate is considered. This problem has the same boundary conditions given in Eq. (3), except for the specification on M. The solution of Eq. (9) gives

$$w(r) = \frac{P}{2\pi G t_h} K_0\left(\sqrt{\frac{\chi}{Gt_h}}\, r\right) \tag{10}$$

Note that Eq. (10) is non-linear in G. Using an appropriate shear modulus G, Shmatkov[7] showed that Eq. (10) fits the measured deflected shape of a floating ice plate close to the melting temperature.

3. DEFORMATION DUE TO BENDING AND SHEAR STRESSES

The experimental evidence suggestes that ice plates have two regimes of short duration deformation. At temperatures close to the melting point the ice plate deforms in shear; at lower remperatures, the plate behaves predominantly in bending. Besides this thermal influence on the ice plate, the simulations conducted in Elvin[9] also indicate that the transverse shear modulus can increase with plate thickness. It is postulated that ice plates in between limiting conditions deflect due to bending and due to shear.

Plates that exhibit both bending and shear deformations are usually referred to as thick plates. Reissner's plate theory can be used to describe thick plate behavior. The governing differential equation of Reissner's thick plates on elastic foundations is

$$D\nabla^4 w + \frac{D}{Gt_h}\nabla^2(q - \chi w) + \chi w = q \tag{11}$$

In Eq. (11), the shear deformation is not due to the thickness of the ice plate as in the standard Reissner plate theory, but due to the reduced transverse shear modulus, G. This reduction in G is believed to be due to grain boundary deformation and due to grain coarsening[9].

Considering once again a floating ice plate subject to a point load, P at $r = 0$, the boundary conditions on Eq. (11) are

$$\begin{aligned} w = \phi = M &= 0 \qquad at \qquad r = \infty \\ \phi &= 0 \qquad at \qquad r = 0 \\ Q &= \lim_{r \to 0}\left(\frac{P}{2\pi r}\right) \end{aligned} \tag{12}$$

where ψ is the rotation of a line initially normal to the neutral surface and is given by

$$\psi = \frac{dw}{dr} - \frac{Q}{Gt_h} \tag{13}$$

Defining two parameters, α and β:

$$\begin{aligned} \cos(\alpha) &= \frac{\sqrt{Dx}}{2Gt_h} \\ \beta^2 &= \frac{\sqrt{Dx}}{2Gt_h} + \sqrt{\frac{Dx}{4(Gt_h)^2} - 1} \end{aligned} \tag{14}$$

the solution[11] to Eq. (11) with the boundary conditions in Eq. (12) is

If $\frac{\sqrt{Dx}}{2Gt_h} \leq 1$:

$$w(r) = \frac{P}{2\pi}\left[\frac{1}{Gt_h}\frac{\sqrt{Dx}}{\sqrt{4(Gt_h)^2 - Dx}} - \frac{\lambda^2}{D}\frac{2Gt_h}{\sqrt{4(Gt_h)^2 - Dx}}\right] Im\left(K_0\left(\frac{r}{\lambda}e^{i\alpha/2}\right)\right) + \frac{P}{2\pi Gt_h} Re\left(K_0\left(\frac{r}{\lambda}e^{i\alpha/2}\right)\right)$$

If $\frac{\sqrt{Dx}}{2Gt_h} > 1$:

$$\begin{aligned} w(r) = \frac{P}{2\pi}\frac{\beta^2\lambda^2}{1-\beta^4}\Bigg[&\left(\frac{1}{D} - \frac{x}{(Gt_h)^2} + \frac{1}{\lambda^2\beta^2}\frac{1}{Gt_h}\right)K_0\left(\frac{r}{\lambda}\beta\right) \\ &+ \left(-\frac{1}{D} + \frac{x}{(Gt_h)^2} - \frac{\beta^2}{\lambda^2}\frac{1}{Gt_h}\right)K_0\left(\frac{r}{\lambda}\frac{1}{\beta}\right)\Bigg] \end{aligned} \tag{15}$$

Note that the solution of a shear and bending plate is not a simple superposition of the solutions from a bending plate(Eq. (4)) and a shear plate(Eq. (10)). This is due to the shear and bending plate resting on an elastic foundation being structurally indeterminate.

When the transverse shear modulus is high, *i.e.* $(Gt_h)^2 \gg Dx$, expected in thicker plates at lower temperatures, the solution in Eq. (15) tends to the bending plate solution, Eq. (4). At the other limit when $(Gt_h)^2 \ll Dx$, *i.e.* when G is very low, Eq. (15) tends to the shear plate solution, Eq. (10).

4. FITTING ELASTIC MODULI TO MEASURED TRANSVERSE DATA

The static elastic moduli of floating ice plates have been determined from deflection measurements due to applied transverse loads. Here the ice plate is assumed to deform both in shear and in bending. The in-plane moduli E and the transverse moduli G of the plate are chosen so that the solution to Eq. (11) best fits the measured load-deflection data. The optimum E, G pair is chosen by the method of least squres. The sum of the squre of the error, e_r^2, between the analytical deflected shape, $\omega(r)$, given by the solution of Eq. (11), and the measured deflection at distance r_i, $\omega^M(r_i)$, is set up:

$$e_r^2 = \sum_{i=1}^{N}(w(r_i) - w^M(r_i))^2 \tag{16}$$

where the summation is carried over the N measured points. The error is minimized by requiring the error to be stationary with respect to E and G:

$$\nabla e_r^2 = 0 \quad \text{or} \quad \left[\frac{\partial e_r^2}{\partial E} \; \frac{\partial e_r^2}{\partial G}\right]^T = 0 \tag{17}$$

Eq. (17) results in two non-linear simultaneous equations in E and G, and these equations are solved by Newton-Raphson iteration.

5. ELASTIG MODULI OF A THIN ICE PLATE - RESULTS AND DISCUSSION

5.1 Simply supported circular plate loaded by a ring load

The calculation of the Young's modulus and the transverse shear modulus requires deflection measurements at several points on the ice sheet at different distances from the load. Tinawi and Gagnon[8] tested thin circular plates of columnar grained S2 sea ice resting only on a ring support (no elastic foundation, κ=0) and loaded by a ring load. The solution[10] of a plate deforming in bending and in shear, simply supported at r=$L/2$ and subject to a ring load (r=$b/2$) of total magnitude P is

$$\omega(r) = \frac{Pb}{8D}\left[\left(r^2+\frac{b^2}{4}\right)\ln\left(\frac{2r}{L}\right)+\left(\frac{L^2}{4}-r^2\right)\frac{L^2(3+\nu)+b^2(\nu-1)}{2L^2(1+\nu)}\right]-\frac{Pb}{2Gt_h}\ln\left(\frac{2r}{L}\right) \text{ if } r\geq b/2$$

$$\tag{18}$$

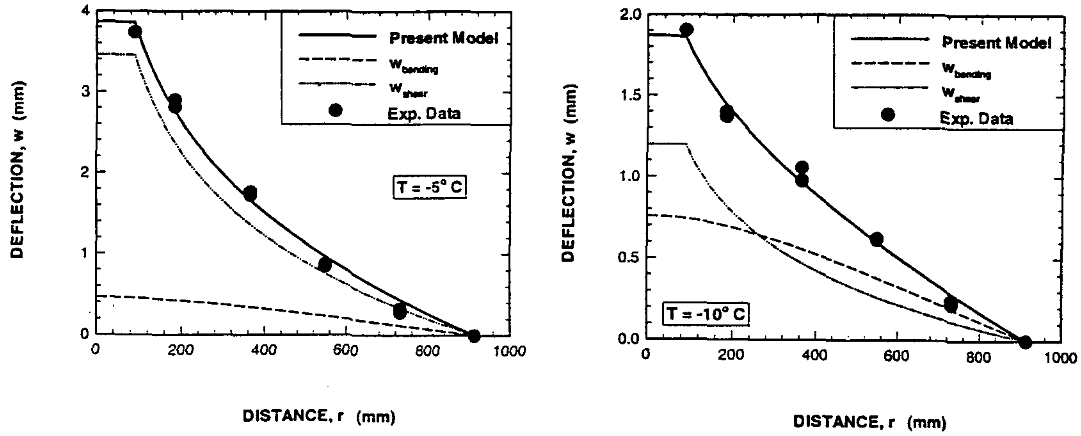
$$\omega(r) = \frac{Pb}{8D}\left[\left(r^2+\frac{b^2}{4}\right)\ln\left(\frac{b}{L}\right)+(L^2-b^2)\frac{L^2(3+\nu)+4r^2(\nu-1)}{8L^2(1+\nu)}\right]-\frac{Pb}{2Gt_h}\ln\left(\frac{b}{L}\right) \text{ if } r< b/2$$

Note that these equations are linear in $1/E$ and $1/G$. Hence the best fit of the data is obtained by linear least squares. The moduli are calculated by substituting Eq. (18) into Eq. (16) and minimizing the error with Eq. (17). A Poisson ratio of ν=0.33 is assumed.

The results for two plates at -5°C and -10°C are summarized in Table 1. The corresponding best fit deflected shapes together with the data of Tinawi and Gagnon[8] are plotted in Fig. 2. From Table 1 and Fig. 2 it can be seen that as the temperature drops, the average Young's modulus of the plate remains approximately constant (falling only by 9%) but the shear modulus increases 3.6 times. The increase in G is expected to be due to the stiffening of the grain boundary. The contribution of shear deformation to the overall deformation falls from 90% to 65% as the temperature decreases from -5°C to -10°C.

Table 1: Elastic moduli fit to the data of Tinawi and Gagnon[8]

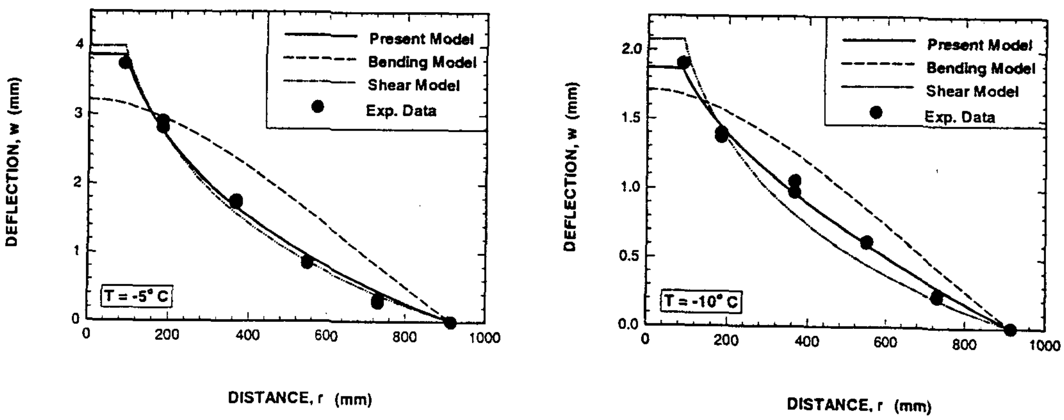
T (°C)	P (N)	t_h (mm)	E (GPa)	G (MPa)	Coeff. of Correlation
-5	4577	107	3.9	4.6	0.976
-10	5239	98	3.56	16.6	0.989



(a) (b)

Figure 2: Fit of thin ice plate deflection data of Tinawi and Gagnon[8] at (a) -5°*C*; (b) -10°*C*. The bending and shear contributions are shown.

Figure 2 shows that the bending and shear deformation curves fit the data well and the coefficient of correlation are high. For comparison, the data is also fitted by assuming only bending and only shear deformations, as shown in Fig. 3. The resulting moduli are: at T=-5°C, E=0.563GPa, G=4MPa; at T=-10°C, E=1.58GPa, G=9.6MPa. Notice that now both E and G are temperature dependent. The Young's moduli are low. Further, the bending only deformation curves do not resemble the shape of the data. Hence these plates cannot be modeled by bending plates.



(a) (b)

Figure 3: Best fit of deflection data of Tinawi and Gagnon[8] by (i) bending and shear medel; (ii) bending only model; and (iii) shear only model, at (a) -5°*C*; (b) -10°*C*.

6. CONCLUSION

This paper examine the effect of including both shear and bending deformations in the behavior of thin elastic ice plates. The in-plane Young's modulus, E, and the transverse shear modulus, *G*, are obtained by fitting an analytical expression for the deflection to experimental data. Results show that thin ice plates deform predominantly in shear, even at temperature lower than melting point. The work presented here shows the importance of accounting for both the bending and shear deformations. The behavior of thin model ice sheets in bending and in shear provides an answer to two concerns raised by Timco[4]. Namely: (i) why the in-plane Young's modulus of thin model ice sheets is so low? and (ii) why a higher Young's modulus is obtained when fitting deflection data points further away from the loading? To emphasize the point: previously, the Young's modulus was calculated assuming only bending behavior while the model ice plates deform predominantly in shear.

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