FUZZY WEAKLY OPEN MAPPINGS AND FUZZY RS-COMPACT SETS IN FUZZY TOPOLOGICAL SPACES

JIN HAN PARK*, MI JUNG SON** AND BU YOUNG LEE**

1. INTRODUCTION AND PRELIMINARIES

Two types of fuzzy almost open mappings between fuzzy topological spaces were introduced by Nanda [10] and by Ganguly and Saha [5], which call them f.a.o.N. and f.a.o.G. respectively. Mukherjee and Sinha [9] showed that f.a.o.N. and f.a.o.G. mappings are independent of each other and investigated similarities and interrelations between these mappings.

In this paper we define fuzzy weakly open mappings which is weaker than f.a.o.N., and show that fuzzy weakly open and f.a.o.G. mappings are also independent notions and investigate mutual interrelations among them. Finally, in section 3, we introduce and discuss the concept of fuzzy RS-compact sets in fuzzy topological space.

Throughtout this paper X means fuzzy topological space (for short, fts) in Chang's [2] sence. For a fuzzy set X in a fts X, the notations $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$ and $1 - A$ will respectively stand for the fuzzy closure, fuzzy interiorand complement of A. By 0_X and 1_X we will means the constant fuzzy sets taking on respectively the values 0 and 1 on X.

Definition. Let A be a fuzzy set in a fts X. Then A is said to be

(a) fuzzy regularly open [1] if $\mathrm{Int}(\mathrm{Cl}(A)) = A$,

(b) fuzzy regularly closed [1] if $\mathrm{Cl}(\mathrm{Int}(A)) = A$,

(c) fuzzy semiopen [1] if there exists a fuzzy open set U such that $U \leq A \leq \mathrm{Cl}(U)$,

(d) fuzzy regular semiopen [6] if there exists a fuzzy regular open set U such that $U \leq A \leq \mathrm{Cl}(A)$.

2. FUZZY WEAKLY OPEN MAPPINGS

Definition 2.1. Let $f : X \to Y$ be a mapping. Then f is said to be

(a) fuzzy weakly open if $f(U) \leq \mathrm{Int}(f(\mathrm{Cl}(U)))$ for each fuzzy open set U in X.

(b) fuzzy almost open (briefly, f.a.o.G.) [4] if $f^{-1}(\mathrm{Cl}(V)) \leq \mathrm{Cl}(f^{-1}(V))$ for each fuzzy open set V in Y.

(c) fuzzy almost open (briefly, f.a.o.N.) [10] if the image of every fuzzy regularly open set is fuzzy open.

Theorem 2.1 [8]. *A mapping $f : X \to Y$ is f.a.o.G. if and only if $f(A) \leq Int(Cl(f(A)))$ for each fuzzy open set A in X.*

Theorem 2.2. *A mapping $f : X \to Y$ is f.a.o.N. if and only if $f(A) \leq Int[f(Int(Cl(A)))]$ for each fuzzy open set A in X.*

It is obvious that every fuzzy a.o.N. mapping is fuzzy weakly open. But the converse is not true as the following Example 2.1 shows. Also Example 3.3 in [8] and Example 2.2 show that fuzzy weakly open and fuzzy a.o.G. mappings are independent notions.

Example 2.1. Let $X = \{a, b, c\}$, $\tau = \{1_X, 0_X, U_1, U_2, U_1 \cup U_2, U_1 \cap U_2\}$, and $\sigma = \{1_X, 0_X, V_1, V_2, V_1 \cup V_2, V_1 \cap V_2\}$, where U_1, U_2, V_1 and V_2 are fuzzy sets in X, respectively, defined as follows:

$$U_1(a) = 0.4,\ U_1(b) = 0.7,\ U_1(c) = 0.2,$$
$$U_2(a) = 0.3,\ U_2(b) = 0.1,\ U_2(c) = 0.6,$$
$$V_1(a) = 0.5,\ V_1(b) = 0.8,\ V_1(c) = 0.3,$$
$$V_2(a) = 0.4,\ V_2(b) = 0.2,\ V_2(c) = 0.7.$$

Consider the identity mapping $f : (X, \tau) \to (Y, \sigma)$. Then f is fuzzy weakly open but not f.a.o.G.

Example 2.2. Let $X = \{a, b, c\}$, $Y = \{x, y, z\}$, and $\tau = \{1_X, 0_X, U\}$, $\sigma = \{1_Y, 0_Y, V_1, V_2\}$, where U, V_1 and V_2 are fuzzy sets in X and Y, respectively, defined as follows:

$$U(a) = 0.5,\ U(b) = 0.3,\ U(c) = 0.2,$$
$$V_1(x) = 0.9,\ V_1(y) = 1,\ \ V_1(z) = 0.7,$$
$$V_2(x) = 0.2,\ V_2(y) = 0.9,\ V_2(z) = 0.3.$$

Consider the mapping $f : (X, \tau) \to (Y, \sigma)$ defined by $f(a) = z, f(b) = x, f(c) = y$. Then f is f.a.o.G. but not fuzzy weakly open.

Theorem 2.3. *A mapping $f : X \to Y$ is fuzzy weakly open if and only if there exists a fuzzy open base $\mathcal{B}$ for the fuzzy topology on X such that for each $B \in \mathcal{B}$, $f(B) \leq Int(f(Cl(B)))$.*

Theorem 2.4. *A mapping $f : X \to Y$ is f.a.o.G. (resp. f.a.o.N.) if and only if there exists a fuzzy open base $\mathcal{B}$ for the fuzzy topology on X such that for each $B \in \mathcal{B}$, $f(B) \leq Int(Cl(f(B)))$ (resp. $f(B) \leq Int[f(Int(Cl(B)))]$).*

Theorem 2.5. *Let $f : X \to Y$ be one-to-one and onto. Then the following statements are equivalent:*

(a) f is a fuzzy weakly open.
(b) $Cl(f(A)) \leq f(Cl(A))$ for each fuzzy open set A in X.
(c) $Cl(f(Int(A))) \leq f(A)$ for each fuzzy closed set A in X.

Definition 2.2 [7]. A fuzzy point x_α is said to be a fuzzy θ-cluster point of a fuzzy set A in a fts X if for each fuzzy open q-nbd U of x_α, $Cl(U)$ is q-coincident with A. The set of all fuzzy θ-cluster points of A is called the fuzzy θ-closure of A and denoted by $Cl_\theta(A)$. A is called fuzzy θ-closed if and only if $A = Cl_\theta(A)$.

Definition 2.3. Let A be any fuzzy set in a fts X. Then fuzzy θ-clsure ($\mathrm{Cl}_\theta(A)$) and fuzzy θ-interior ($\mathrm{Int}_\theta(A)$) of A defined as follows:

$$\mathrm{Cl}_\theta(A) = \cap\{B \mid B \text{ is fuzzy open and } A \leq B\},$$
$$\mathrm{Int}_\theta(A) = \cup\{B \mid B \text{ is fuzzy closed and } B \leq A\}.$$

Theorem 2.6. *If $f : X \to Y$ is fuzzy weakly open mapping, then the following are equivalent:*
(a) $f^{-1}(Cl(B)) \leq Cl_\theta(f^{-1}(B))$ for each fuzzy set B in Y.
(b) $f(Int_\theta(A)) \leq Int(f(A))$ for each fuzzy set A in X.
(c) $f(A) \leq Int(f(A))$ for each fuzzy θ-open set A in X.
(d) For any fuzzy set B in Y and any fuzzy θ-closed set A in X containing $f^{-1}(B)$, there exists a fuzzy closed set C in Y containing B such that $f^{-1}(C) \leq A$.

Theorem 2.7. *If X is fuzzy regular and $f : X \to Y$ is fuzzy weakly open mapping, then f is fuzzy open.*

Theorem 2.8. *If $f : X \to Y$ is f.a.o.G. and if for each fuzzy set U from some open base for the fuzzy topology on X, $Cl(f(U)) \leq f(Cl(U))$, then f is fuzzy weakly open.*

Definition 2.4. A mapping $f : X \to Y$ is called
(a) fuzzy almost continuous [8] if for each fuzzy point x_α of X and each fuzzy open nbd V of $f(x_\alpha)$, $\mathrm{Cl}(f^{-1}(V))$ is a fuzzy nbd of x_α,
(b) fuzzy θ-continuous [7] if for each fuzzy point x_α and each fuzzy open q-nbd V of $f(x_\alpha)$, there exists a fuzzy open q-nbd U of x_α such that $f(\mathrm{Cl}(U)) \leq \mathrm{Cl}(V)$.

Theorem 2.9. *If a mapping $f : X \to Y$ is fuzzy weakly open and fuzzy almost continuous, then f is f.a.o.G.*

Corollary 2.10. *If a mapping $f : X \to Y$ is f.a.o.N. and fuzzy almost continuous, then f is f.a.o.G.*

Theorem 2.11. *Let $f : X \to Y$ be a f.a.o.G. Then f is fuzzy θ-continuous if and only if $Cl_\theta(f^{-1}(V)) = f^{-1}(Cl(V))$ for each fuzzy open set V in Y.*

3. Fuzzy RS-compact sets

Definition 3.1 [6]. A collection $\mathcal{U}$ of fuzzy (resp. fuzzy open, fuzzy semiopen) sets in a fts X is said to be a fuzzy (resp. fuzzy open, fuzzy semiopen) cover of a fuzzy set A in X if $(\cup\mathcal{U})(x) = 1$ for all $x \in \mathrm{Supp}(A)$. A fuzzy cover $\mathcal{U}$ of a fuzzy set A in X is said to have a finite subcover (resp. fuzzy proximate subcover) $\mathcal{U}_0$ for A if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ and $\cup\mathcal{U}_0 \geq A$ (resp. $\cup\{\mathrm{Cl}(U) \mid U \in \mathcal{U}_0\} \geq A$).

Definition 3.2. A fuzzy set A in a fts X is said to be a fuzzy almost compact set (simply, FAC-set) [9] (resp. fuzzy S-closed set (simply, FRS-set) [6]) if every fuzzy open (resp. semiopen) cover of A has a finite fuzzy proximate subcover.

Definition 3.3. A fuzzy set A in a fts X is said to be a fuzzy RS-compact set (simply, FRSC-set) if for every fuzzy regular semiopen cover $\mathcal{U}$, there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $\cup\{\text{Int}(U) \mid U \in \mathcal{U}_0\} \geq A$.

Theorem 3.1. *For a fuzzy set A in a fts X. the following are equivalent:*

(a) A is FRSC-set in X.

(b) For every fuzzy cover $\mathcal{U}$ of A by fuzzy regularly closed sets, there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \leq \cup\{\text{Int}(U) \mid U \in \mathcal{U}_0\}$.

(c) If $\mathcal{U}$ is family of fuzzy regularly open sets having the property that for every finite subcolloection $\mathcal{U}_0$ of $\mathcal{U}$, $[\cap\{Cl(U) \mid U \in \mathcal{U}_0\}](x) > 1 - A(x)$ for some $x \in Supp(A)$, then $\cap\mathcal{U} \neq 0_X$.

(d) For every family $\mathcal{U}$ of fuzzy regularly open sets in X such that $[\cap\{1 - U \mid U \in \mathcal{U}\}]\bar{q}A$, there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $[1 - \cap\{Cl(U) \mid U \in \mathcal{U}_0\}]\bar{q}A$.

(e) For every family $\mathcal{U}$ of fuzzy regular semiopen sets in X such that $[\cap\{1 - U \mid U \in \mathcal{U}\}]\bar{q}A$, there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $[\cap\{Cl(U) \mid U \in \mathcal{U}_0\}]\bar{q}A$.

Theorem 3.2. *A fuzzy set A in a fts X is a FRSC-set in X if and only if for every fuzzy regular semiopen filterbase $\mathcal{B}$ in X, $[\cap\{Int(B) \mid B \in \mathcal{B}] \cap A = 0_X$ implies that there exists a finite subcollection $\mathcal{B}_0$ of $\mathcal{B}$ such that $\cap\{B \mid B \in \mathcal{B}_0\}\bar{q}A$.*

Theorem 3.3. *If A is FRSC-set in a fts X and if any fuzzy filterbase $\mathcal{B}$ in X has the property that for any finitely many members $B_1, B_2, \cdots, B_n$ of $\mathcal{B}$ and for any regular semiopen set $C \geq A$, $(B_1 \cap B_2 \cap \cdots \cap B_n)qC$ holds, then $\mathcal{B}$ has a fuzzy rs-accumulation point [11] in A.*

Theorem 3.4. *Every fuzzy regular semiopen set A in a fuzzy RS-compact sapce is a FRSC-set.*

Corollary 3.5. *If a fuzzy set A is either fuzzy regularly open or fuzzy regularly closed in a fuzzy RS-compact space, then A is a FRSC-set in X.*

Theorem 3.6. *Let $f : X \to Y$ be a f.a.o.G and fuzzy weakly continuous mapping. If A is a FRSC-set in X, then $f(A)$ is a FRSC-set in Y.*

Theorem 3.7. *Let $f : X \to Y$ be a fuzzy weakly open and fuzzy θ-continuous mapping. If A is a FRSC-set in X, then $f(A)$ is a FRSC-set in Y.*

References

[1] K.K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82(1981) 14–32.

[2] C.L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24(1968) 182-190.
[3] J.R. Choi, B.Y. Lee and J.H. Park, On the fuzzy θ-continuous mappings, Fuzzy Sets and Systems 54 (1993) 107-114.
[4] S. Ganguly and S. Saha, A note on semi-open sets in fuzzy topological spaces, Fuzzy Sets and Systems 18(1986) 83-96.
[5] P.P. Ming and L.Y. Ming, Fuzzy topology I. Neighborhood structure of fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. 76(1980) 571-599.
[6] M.N. Mukherjee and B. Ghosh, On fuzzy S-closed spaces and FSC sets, Bull. Malaysian Math. Soc. 12 (1989) 1-14.
[7] M.N. Mukherjee and S.P. Sinha, On some weaker forms of fuzzy continuous and fuzzy open mappings on fuzzy topological spaces, Fuzzy Sets and Systems 32(1989) 103-114.
[8] M.N. Mukherjee and S.P. Sinha, Irresolute and almost functions between fuzzy topological spaces, Fuzzy Sets and Systems 29(1989) 381-388.
[9] M.N. Mukherjee and S.P. Sinha, Almost compact fuzzy sets in fuzzy toplogical spaces Fuzzy Sets and Systems 38 (1990) 389-396.
[10] S. Nanda, On fuzzy topological spaces, Fuzzy Sets and Systems 19(1986) 193-197.
[11] Y.B. Park, S.J. Cho and J.H. Park, Fuzzy RS-compact topological spaces, Fuzzy Sets and Systems 77 (1996) 241—246.
[12] J.H. Park, B.Y. Lee and J.R. Choi, Fuzzy θ-connectedness, Fuzzy Sets and Systems 59 (1993) 237-244.
[13] R. Srivastava, S.N. Lal and A.K. Srivastava, Fuzzy Hausdorff topological spaces, J. Math. Anal. Appl. 81(1981) 499-506.
[14] T.H. Yalvac, Fuzzy sets and functions on fuzzy topological spaces, J. Math. Anal. Appl. 126(1987) 409-423.

* Department of Applied Mathematics, Pukyong National University, Pusan 608-737, Korea

** Department of Mathematics, Dong-A University, Pusan 604-714, Korea