On t-Fuzzy Equivalence Relations

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I. Introduction.

Since the definition of a fuzzy relation from X to Y was introduced , as a fuzzy subset of $X\times Y$, by Zadeh[6], the theory of fuzzy relations was developed by [3, 4, 5]. In [4], Nemitz has studied lattice-valued fuzzy relations assuming values from a Brouwerian lattice. In [5], Sidky defined the t-fuzzy partition on a set and studied its properties.

In this paper we define the notion of t-fuzzy relation on a set and proved some related properties.

II. Preliminaries

Let L denote a linear lattice with the universal bounds $0, 1$ and $t\in L-\{0\}$ throughout this paper .

Definition 2.1. [5]. (1) Let X be a nonempty set and L be a lattice. An L-fuzzy subset A of X is defined to be a mapping $A: X\rightarrow L$.
(2) Let $F(X)=\{A \mid A$ is an L-fuzzy subset of $X\}$.

Definition 2.2. [5] Let A be an L-fuzzy subset of X and $t\in L$. The subset $\overline{A^t}=\{x\in X \mid A(x) > t\}$ of X is called a strong t-level subset of X.

Definition 2.3. [5] Let X and Y be nonempty sets. A fuzzy relation from X to Y is defined to be an L-fuzzy subset of $X\times Y$.

Definition 2.4. [4] Let X be a nonempty set and $R\in F(X\times Y)$, R is called an L-valued fuzzy equivalence relation on X iff

(1) R is reflexive i.e, $R(x,x)=1$, $^{\forall}x\in X$.

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(2) R is symmetric i.e, $R^{-1}=R$.

(3) R is transitive i.e, $R \circ R \subseteq R$.

Definition 2.5. [5] Let X be a nonempty set. The class $P=\{Y_i \in P(X) \mid i \in I\}$ is called a t-fuzzy partition of X iff

(1) for each $x \in X$, there is $Y_i \in P$ such that $Y_i(x) > t$.

(2) for each $Y_i \in P$, there is $x \in X$ such that $Y_i(x) > t$.

(3) if $i \neq j$, then $Y_i(x) \wedge Y_j(x) \leq t$, $\forall x \in X$.

III. Main Results

Definition 3.1. Let $R \in F(X \times X)$. R is called a t-fuzzy equivalence relation on X iff

(1) R is t-reflexive i.e. $R(x,x) > t$, $\forall x \in X$.

(2) $R^{-1}=R$.

(3) $R \circ R \subseteq R$.

Theoreme 3.2. Let $R \in F(X \times X)$ be a t-fuzzy equivalence relation on X. Then $\overline{R^t}=\{(x,y) \mid R(x,y) > t\}$ is an ordinary equivalence relation on X.

Proof. Firstly, for each $x \in X$, since R is t-reflexive, $R(x,x) > t$, and so $(x,x) \in \overline{R^t}$, which implies $\overline{R^t}$ is reflexive.

Secondly, let $(x,y) \in \overline{R^t}$. Then $R(x,y) > t$. Since R is symmetric, $R(y,x)=R^{-1}(x,y) > t$, and so $(y,x) \in \overline{R^t}$, which implies $\overline{R^t}$ is symmetric. Finally, let $(x,y) \in \overline{G^t}$, and $(y,z) \in \overline{R^t}$. Then $R(x,y) > t$ and $R(y,z) > t$. By transitivity of R, $R(x,z) > t$, and so $(x,z) \in \overline{R^t}$, which implies $\overline{R^t}$ is transitive. Therefore, $\overline{R^t}$ is an ordinary equivalence relation on X.

Theorem 3.3. Let $R \in F(X \times X)$ is a t-fuzzy equivalence relation on X, and let $[x]_{\overline{R^t}}=\{y \in X \mid R(x,y) > t\}$, $x \in X$

Then $\{[x]_{\overline{R^t}} \mid x \in X\}$ is an ordinary partition of X.

Proof. For each $x \in X$, since R is t-reflexive, $R(x,x) > t$ and so $x \in [x]_{\overline{R^t}}$, which implies $X \subseteq \bigcup_{x \in X} [x]_{\overline{R^t}}$.

We claim that $[x]_{\overline{R^t}} = [y]_{\overline{R^t}}$ if $[x]_{\overline{R^t}} \cap [y]_{\overline{R^t}} \neq \emptyset$. Suppose that $[x]_{\overline{R^t}} \cap [y]_{\overline{R^t}} \neq \emptyset$, then there exists $z \in X$ such that $z \in [x]_{\overline{R'}} \cap [y]_{\overline{R'}}$. This implies that $R(x, z) > t$ and $R(y, z) > t$.

By transitivity of R, $R(x, y) > t$. If $z' \in [x]_{\overline{R'}}$, then $R(x, z') > t$. Combining $R(x, y) > t$ and $R(z', x) > t$, which implies $R(z', y) > t$, hence $[x]_{\overline{R'}} \subseteq [y]_{\overline{R'}}$. Similiarly, interchanging the roles of x and y, we get that $[y]_{\overline{R'}} \subseteq [x]_{\overline{R'}}$. Therefore, $[x]_{\overline{R'}} = [y]_{\overline{R'}}$.

Definition 3.4. Let $R \in F(X \times X)$ be a t-fuzzy equivalence relation on X. For each $x \in X$, we define a fuzzy subset $P_{[x]_{\overline{R'}}}: X \to I$ by

$$P_{[x]_{\overline{R'}}}(z) = \begin{cases} \bigvee_{R(x,z)>t} R(x, z) & \text{if } z \in [x]_{\overline{R'}} \\ 0 & \text{if } z \notin [x]_{\overline{R'}} \end{cases}$$

Theorem 3.5. If $[x]_{\overline{R'}} \cap [y]_{\overline{R'}} \neq \emptyset$, then $P_{[x]_{\overline{R'}}}(z) = P_{[y]_{\overline{R'}}}(z)$, $z \in X$.

Proof. Let $y \in [x]_{\overline{R'}}$. We show that $P_{[x]_{\overline{R'}}}(z) = P_{[y]_{\overline{R'}}}(z)$, $z \in X$. If $z \in [x]_{\overline{R'}}$, then $R(x, z) > t$. By transivity of R, $z \in [y]_{\overline{R'}}$ which implies $P_{[x]_{\overline{R'}}}(z) = P_{[y]_{\overline{R'}}}(z)$. Otherwise, since $y \in [x]_{\overline{R'}}$, $z \notin [y]_{\overline{R'}}$, which implies $P_{[x]_{\overline{R'}}}(z) = 0 = P_{[y]_{\overline{R'}}}(z)$. Therefore, $P_{[x]_{\overline{R'}}}(z) = P_{[y]_{\overline{R'}}}(z)$ for all $z \in X$.

Theorem 3.6. $P = \{P_{[x]_{\overline{R'}}} \mid x \in X\}$ is a t-fuzzy partition of X.

Proof. Firstly, for each $z \in X$, Since R is t-reflexive $R(z, z) > t$, which implies $P_{[z]_{\overline{R'}}}(z) \geqq R(z, z) > t$.

Secondly, for each $P_{[z]_{\overline{R'}}} \in P$, we have $P_{[z]_{\overline{R'}}}(z) > t$.

Finally, we show that $[x]_{\overline{R'}} \cap [y]_{\overline{R'}} = \emptyset$ implies $P_{[x]_{\overline{R'}}}(z) \wedge P_{[y]_{\overline{R'}}}(z) \leqq t$, $z \in X$. If $z \in [x]_{\overline{R'}}$, since $\{[x]_{\overline{R'}} \mid x \in X\}$ is an ordinary partition of X, then $z \notin [y]_{\overline{R'}}$, and so $P_{[y]_{\overline{R'}}}(z) = 0$, which implies $P_{[x]_{\overline{R'}}}(z) \wedge P_{[z]_{\overline{R'}}}(z) = 0$. If $z \in [y]_{\overline{R'}}$, then $z \notin [x]_{\overline{R'}}$, and so $P_{[x]_{\overline{R'}}}(z) = 0$, which implies $P_{[x]_{\overline{R'}}}(z) \wedge P_{[y]_{\overline{R'}}}(z) = 0$. If $z \notin [x]_{\overline{R'}}$ and $z \notin [y]_{\overline{R'}}$, then $P_{[x]_{\overline{R'}}}(z) = 0 = P_{[y]_{\overline{R'}}}(z)$, and so $P_{[x]_{\overline{R'}}}(z) \wedge P_{[y]_{\overline{R'}}}(z) = 0$. In which case, we have $P_{[x]_{\overline{R'}}}(z) \wedge P_{[y]_{\overline{R'}}}(z) \leqq t$ for all $z \in X$.

Theorem 3.7. Let $P = \{P_i \in F(X) \mid i \in I\}$ be a t-fuzzy partition of X. Then there exists the correspondent t-fuzzy equivalence relation on X.

Proof. Define a fuzzy subset $R: X\times X \to I$ by

$$R(x,y) = \vee_i P_i(x) \wedge P_i(y), \quad (x,y) \in X\times X.$$

We show that R is a t-fuzzy equivalence relation on X. Firstly, let $x \in X$. Since P is a t-fuzzy partition of X, there exists a $P_i \in P$ such that $P_i(x) > t$. This implies that $R(x,x) = \vee_i P_i(x) \wedge P_i(x) = \vee_i P_i(x) \geq P_i(x) > t$. Therefore, R is t-reflexive.

Secondly, by definition, $R(x,y) = R(y,x)$.

Finally,

$$\begin{aligned} R \circ R(x,y) &= \vee_{z\in X} R(x,z) \wedge R(z,y) \\ &= \vee_{z\in X} (\vee_i (P_i(x) \wedge P_i(z)) \wedge \vee_i (P_i(z) \wedge P_i(y))) \\ &\leqq \vee_{z\in X} (\vee_i (P_i(x) \wedge P_i(z) \wedge P_i(y)) \\ &\leqq \vee_i \vee_{z\in X} (P_i(x) \wedge P_i(y)) \\ &= \vee_i \ (P_i(x) \wedge P_i(y)) = R(x,y) \quad \text{for all } (x,y) \in X\times X. \end{aligned}$$

Therefore, R is transitive. Consequently, R is a t-equivalence relation on X.

Theorem 3.8. Let R be a t-fuzzy equivalence relation on X and let $t > s$, $0 \leq s < 1$. Then R is a s-fuzzy equivalence relation on X, and $P_t \subseteq P_s$, where $P_t = \{P_{[x]_{\overline{R^t}}} \mid x \in X\}$, $P_s = \{P_{[x]_{\overline{R^s}}} \mid x \in X\}$.

Proof. To show that R is a s-fuzzy equivalence relation on X, it suffices to show that R is s-reflexive. For each $x \in X$, since R is t-reflexive, $R(x,x) > t$, which implies $R(x,x) > s$. Therefore, R is s-reflexive. The rest is clear. If $y \in [x]_{\overline{R^t}}$, then $R(x,y) > t$. Since $t > s$, which implies $R(x,y) > s$ and so $y \in [x]_{\overline{R^s}}$, therefore, $[x]_{\overline{R^t}} \subseteq [x]_{\overline{R^s}}$. Consequently, we have $P_t \subseteq P_s$.

REFERENCES

1. G. Birkhoff, Lattice theory, A. M. S. Providence, Rhode Ishand, 1967.

2. D. Dubois and H. Prade, Fuzzy Set and System-Theory and Applications (Academic Press, New York, 1980).

3. V. Murali, Fuzzy equivalence relations, Fuzzy Sets and Systems 30 (1989) 155-163

4. W. C Nemitz, Fuzzy relations and fuzzy functions, Fuzzy Sets and Systems 19 (1986) 177-191.

5. F. I. Sidky, t-fuzzy mapping, Fuzzy Sets and Systems 76 (1995) 387-393.

6. L. A. Zadeh, Fuzzy Sets, Inform and Control 8 (1965) 338-353