SEMI-SEPARATION AXIOMS IN FUZZY TOPOLOGICAL SPACES

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ABSTRACT

In this paper, certain fuzzy semi-separation axioms are studied in terms of the notions of quasi-coincidence, fuzzy semi-q-neigborhoods and fuzzy semi-θ-closure operators. Fuzzy semi-T_2, fuzzy semi-Urysohn and fuzzy s-regular spaces are defined, and fuzzy spaces satisfying these axioms are characterized.

1. INTRODUCTION AND PRELIMINARIES

Using the concept of fuzzy set, Chang [2] first introduced fuzzy topological space (fts, for short). Subsequently many authors [1,5,6,9-12] continued the investigation of such spaces. The concepts of fuzzy semiopen and semiclosed sets were first introduced in a fts by Azad [1]. Using these concepts, Ghosh [4] introduced fuzzy semi-T_i spaces ($i = 0, 1, 2$) and studied these spaces under fuzzy semi-continuity. In this paper, we study certain fuzzy semi-separation axioms in terms of the notions of quasi-coincidence, fuzzy semi-q-neighborhoods and fuzzy semi-θ-closure operators. Fuzzy semi-T_2, fuzzy semi-Urysohn and fuzzy s-regular spaces are defined, and fuzzy spaces satisfying these axioms are characterized.

We now explain some concepts and notations to be used frequently in the sequel. Throughout this paper, by (X, τ) (or simply X) we mean a fuzzy topological space in Chang's [2] sense. A fuzzy point in X with support $x \in X$ and value α $(0 < \alpha \leq 1)$ is denoted by x_α. For a fuzzy set A in X, $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$, $1 - A$ and $(A)_0$ will respectively denote the closure, interior, complement and support of A. A fuzzy set A in X is said to be q-coincident with a fuzzy set B, denoted by AqB, if there exists $x \in X$ such that $A(x) + B(x) > 1$ [10]. It is known in [10] that $A \leq B$ if and only if A and $1 - B$ are not q-coincident, denoted by $A\bar{q}(1 - B)$. A fuzzy set A in a fts X is called fuzzy semiopen (semiclosed) if there exists a fuzzy open (resp. fuzzy closed) set B such that $B \leq A \leq \mathrm{Cl}(B)$ (resp. $\mathrm{Int}(B) \leq A \leq B$) [1].

A fuzzy set A in X is said to be a fuzzy semi-nbd [4] (fuzzy semi-q-nbd [4]) of x_α if there exists a fuzzy semiopen set U in X such that $x_\alpha \in U \leq A$ (resp. $x_\alpha qU \leq A$). The fuzzy semi-closure of a fuzzy set A in X, denoted by $\mathrm{sCl}(A)$, is the union of all fuzzy point x_α such that every fuzzy semiopen semi-q-nbd of x_α is q-coincident with A. equivalent $\mathrm{sCl}(A)$ is the intersection of all fuzzy semiclosed sets containing A [4]. A is fuzzy semiclosed if and only if $A = \mathrm{sCl}(A)$. The union of all fuzzy semiopen sets contained in fuzzy set A in X is called the fuzzy semi-interior of A, denoted by $\mathrm{sInt}(A)$. A fuzzy point x_α is said to be a fuzzy semi-θ-cluster point [11] of a fuzzy set A in X if the fuzzy semi-closure of every fuzzy semiopen semi-q-nbd of x_α is q-coincident with A. The union of all fuzzy semi-θ-cluster points of A is called the fuzzy semi-θ-closure of A is denoted $[A]_{s\text{-}\theta}$ [11]. A fuzzy set A is called fuzzy semi-θ-closed [10] if $A = [A]_{s\text{-}\theta}$, and complement of a fuzzy semi-θ-closed set is fuzzy semi-θ-open. Obviously, every fuzzy semi-θ-closed set is fuzzy semiclosed. For other definitions and results, not explained here, we refer to the standard papers [4,7,9,10].

Lemma 1.1 [7]. *If A is any fuzzy set and B is a fuzzy semiopen set in a fts X such that $A\bar{q}B$, then $sCl(A)\bar{q}B$.*

Lemma 1.2 [7]. *For a fuzzy semiopen set A in a fts X, $sCl(A) = [A]_{s\text{-}\theta}$.*

Lemma 1.3. *For any fuzzy subset A in a fts X,*

$$[A]_{s\text{-}\theta} = \bigwedge\{[U]_{s\text{-}\theta} \mid U \text{ is fuzzy semiopen and } A \leq U\}.$$

2. Fuzzy semi-T$_2$ axioms

Definition 2.1 [4]. A fts X is said to be fuzzy G-semi-T$_2$ if for any two distinct fuzzy points x_α and y_β, the following conditions are satisfied:

(a) when $x \neq y$, x_α and y_β have fuzzy semiopen semi-nbds which are not q-coincident;

(b) when $x = y$ and $\alpha < \beta$ (say), x_α has a fuzzy semiopen semi-nbd U and y_β has a fuzzy semiopen semi-q-nbd V such that $U\bar{q}V$.

Definition 2.2. A fts X is said to be fuzzy semi-T$_2$ if every fuzzy point is fuzzy semi-θ-closed.

Theorem 2.3. *A fts X is fuzzy semi-T_2 if and only if for any two distinct fuzzy points x_α and y_β: when $x \neq y$, there exist fuzzy semiopen sets U_1, U_2, V_1 and V_2 such that $x_\alpha \in U_1$, $y_\beta q V_1$, $U_1 \bar{q} V_1$ and $x_\alpha q U_2$, $y_\beta \in V_2$, $U_2 \bar{q} V_2$; when $x = y$ and $\alpha < \beta$ (say), there exist fuzzy semiopen sets U and V such that $x_\alpha \in U$, $x_\beta q V$ and $U\bar{q}V$.*

Theorem 2.4. *A fts X is fuzzy semi-T_2 if and only if for every fuzzy point x_α, $x_\alpha = \bigwedge\{sCl(U) \mid U$ is fuzzy semiopen semi-nbd of $x_\alpha\}$.*

In the first case (when $x \neq y$) of necessary part of the Theorem 2.3, fuzzy semiopen semi-nbd can be replaced by fuzzy semiopen semi-q-nbd.

Theorem 2.5. *If a fts X is fuzzy semi-T_2, then for every two distinct fuzzy points x_α and y_β in X: when $x \neq y$, x_α and y_β have fuzzy semiopen semi-q-nbds U and V respectively such that $U\bar{q}V$; when $x = y$ and $\alpha < \beta$ (say), x_α has a fuzzy semiopen semi-nbd U and y_β has a fuzzy semiopen semi-q-nbd V such that $U\bar{q}V$.*

As a partial converse of the above theorem we have the following:

Theorem 2.6. *If in a fts X, the necessary condition of the above Theorem 2.5 hold, then every fuzzy point x_α ($\alpha \le \frac{1}{2}$) is fuzzy semi-θ-closed, and $x_\beta \notin [x_\alpha]_{s\text{-}\theta}$ if $\beta > \alpha$, for any $\alpha \in (0, 1)$.*

Theorem 2.7. *A necessary condition for a fts X to be fuzzy semi-T_2 is that for any fuzzy set A in X, $A = A^*$, where $A^* = \bigvee\{x_\alpha \mid$ there exists $y \in ([x_\alpha]_{s\text{-}\theta})_0$ such that $([x_\alpha]_{s\text{-}\theta})(y) \le A(y)\}$.*

Remark. The converse of the above theorem is true if it would so happen that for two fuzzy points x_α and y_β, $x_\alpha \in [y_\beta]_{s\text{-}\theta}$ if and only if $y_\beta \in [x_\alpha]_{s\text{-}\theta}$. But we observe that this is not true (see Example 2.8). Nevertheless, it is true that if for every fuzzy point x_α in X, $x_\alpha = (x_\alpha)^*$, then for any fuzzy point y_β,

$$y_\beta \notin [y_\alpha]_{s\text{-}\theta} \quad \text{if } \beta > \alpha.$$

In fact, since $y_\alpha \in (y_\alpha)^* = y_\alpha$, there exists $z \in ([y_\alpha]_{s\text{-}\theta})_0$ such that $([y_\alpha]_{s\text{-}\theta})(z) \le (y_\alpha)(z)$. Then $z = y$ so that $([y_\alpha]_{s\text{-}\theta})(y) = \alpha$.

Example 2.8. Let $X = \{a, b\}$. Consider the fuzzy topology $\tau = \{1_X, 0_X, A\}$, where $A(a) = \frac{1}{3}$ and $A(b) = 0$. Then 1_X, 0_X and C are fuzzy semiopen set in (X, τ), where C is fuzzy set in X defined by $\frac{1}{3} \le C(a) \le \frac{2}{3}$ and $0 \le C(b) \le 1$. Now consider two fuzzy points $a_{\frac{1}{12}}$ and $b_{\frac{4}{5}}$. Then 1_X is only fuzzy semiopen semi-q-nbd of $a_{\frac{1}{12}}$. Hence $a_{\frac{1}{12}} \in [b_{\frac{4}{5}}]_{s\text{-}\theta}$. But we consider fuzzy set D defined by $D(a) = \frac{1}{3}$ and $D(b) = \frac{1}{4}$. Then D is fuzzy semiopen semi-q-nbd of $b_{\frac{4}{5}}$ and $a_{\frac{1}{12}} \bar{q} sCl(D)$. Hence $b_{\frac{4}{5}} \notin [a_{\frac{1}{12}}]_{s\text{-}\theta}$.

Theorem 2.9. *A fts X is fuzzy G-semi-T_2 if and only if X is fuzzy semi-T_2 and for $x, y \in X$ with $x \ne y$, there exists a fuzzy semiopen semi-nbd U of x_α such that $y \notin (sCl(U))_0$.*

Theorem 2.10. *If X is fuzzy semi-T_2, then for any two distinct fuzzy points x_α and y_β with $x \ne y$ or else $x = y$ and $\frac{1}{2} < \alpha < \beta$, there exist fuzzy semiopen sets U and V such that $x_\alpha q U$, $y_\beta q V$ and $U \wedge V \le \frac{1}{2}$.*

The converse of above theorem is not true as following example shows.

Example 2.11. Let X consist of a single point a and the fuzzy topology τ on X consist of all fuzzy sets such that $A(a) \le \frac{1}{3}$ together with 1_X. Consider the fuzzy points $a_{\frac{7}{10}}$ and $a_{\frac{8}{10}}$. Then 1_X is the only fuzzy semiopen semi-nbd of $a_{\frac{7}{10}}$ and thus there cannot exist a fuzzy semiopen semi-nbd U of $a_{\frac{7}{10}}$ and a fuzzy semiopen semi-q-nbd V of $a_{\frac{8}{10}}$ such that $U \bar{q} V$. Hence by Theorem 2.3, (X, τ) is not fuzzy semi-T_2. But for any two fuzzy points a_α and a_β with $\frac{1}{2} < \alpha < \beta$, putting $U = V = a_{\frac{1}{2}}$, then U and V are fuzzy semiopen semi-q-nbd of a_α and a_β respectively such that $U \wedge V \le \frac{1}{2}$.

In [4], Ghosh showed that a fts X is fuzzy semi-T_1 if and only if every fuzzy point in X is fuzzy semiclosed. Since every fuzzy semi-θ-closed set is fuzzy semiclosed, it follows that a fuzzy semi-T_2 space is fuzzy semi-T_1. But the converse is not necessarily true is shown by the following example.

Example 2.12. Let X be an infinite set. Suppose τ consists of 0_X and all those fuzzy sets in X whose complements have finite supports. Then (X,τ) is a fts. Clearly, (X,τ) is fuzzy semi-T_1. But, for any fuzzy semiopen set U in X, $U(x) = 1$ for all but finitely many points x of X. Thus any two non-null fuzzy semiopen sets in X are q-coincident. Hence (X,τ) is not semi-T_2.

Definition 2.13 [7]. A fts X is said to be fuzzy s-regular if for each fuzzy point x_α in X and each fuzzy semiopen semi-q-nbd V of x_α, there exists a fuzzy semiopen semi-q-nbd U of x_α such that $\mathrm{sCl}(U) \leq V$.

Theorem 2.14. *For a fts X the following are equivalent:*

(a) X is fuzzy s-regular.

(b) For each fuzzy point x_α in X and each fuzzy semiclosed set F with $x_\alpha \notin F$, there exists a fuzzy semiopen set U such that $x_\alpha \notin sCl(U)$ and $F \leq U$.

(c) For each fuzzy point x_α in X and each fuzzy semiclosed set F with $x_\alpha \notin F$, there exist fuzzy semiopen sets U and V such that $x_\alpha qU$, $F \leq V$ and $U\bar{q}V$.

(d) For each fuzzy set A and each fuzzy semiclosed set F with $A \not\leq F$, there exist fuzzy semiopen sets U and V such that AqU, $F \leq V$ and $U\bar{q}V$.

(e) For each fuzzy set A and each fuzzy semiopen set U with AqU, there exists a fuzzy semiopen set V such that $AqV \leq sCl(V) \leq U$.

Theorem 2.15. *A fts X is fuzzy s-regular if and only if $sCl(A) = [A]_{s\text{-}\theta}$ for each fuzzy set A in X.*

Definition 2.16. A fts X is said to be fuzzy semi-Urysohn if for any two distinct fuzzy points x_α and y_β in X:

(a) When $x \neq y$, there exist fuzzy semiopen sets U_1, U_2, V_1 and V_2 such that $x_\alpha \in U_1$, $y_\beta \mathrm{q} V_1$ and $\mathrm{sCl}(U_1)\bar{\mathrm{q}}\mathrm{sCl}(V_1)$, and $x_\alpha \mathrm{q} U_2$, $y_\beta \in V_2$ and $\mathrm{sCl}(U_2)\bar{\mathrm{q}}\mathrm{sCl}(V_2)$;

(b) When $x = y$ and $\alpha < \beta$ (say), there exist two fuzzy semiopen sets U and V such that $x_\alpha \in U$, $y_\beta \mathrm{q} V$ and $\mathrm{sCl}(U)\bar{\mathrm{q}}\mathrm{sCl}(V)$.

Every fuzzy semi-Urysohn space is fuzzy semi-T_2.

Theorem 2.17. *A fts X is fuzzy semi-Urysohn if and only if for each fuzzy point x_α in X, $x_\alpha = (x_\alpha)^{**}$, where $(x_\alpha)^{**} = \bigwedge\{[[U]_{s\text{-}\theta}]_{s\text{-}\theta} \mid x_\alpha \in U$ and U is fuzzy semiopen$\}$.*

Theorem 2.18. *Every fuzzy s-regular and fuzzy semi-T_2 space is fuzzy semi-Urysohn.*

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