Convergence and Compactness

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1 Introduction

The notion of convergence in a topological space plays an essential role in various areas in mathematics. Particularly using filters or nets it characterizes the notion of compactness in a topological space in terms of cluster points and limit points. In fuzzy topology, various notions of compactness have been studies using fuzzy open sets and functors between fuzzy topological spaces. For examples, quasi-fuzzy compactness, α-compactness, weakly fuzzy compactness, α^*-compactness, strong fuzzy compactness[4] and solid fuzzy compactness[5]. Moreover using prefilters and fuzzy nets the notions of nice compactness [9], ultrafilter compactness[1] and stable fuzzy compactness[6] are defined in a fuzzy topological space. Of course the concepts of prefilter convergence are basic tools to introduce those notions. We note that each notion uses a different type of convergence. On the other hand, there are many other different types of convergence in fuzzy topology. This means that we can define various kinds of compactness in a fuzzy topological space. In this paper, first we introduce an universal way to define a notion of compactness using cluster points of a prefilter and to prove Tychonoff theorem using ultra(maximal) prefilters.

Various kinds of examples are followed as good extentions of compactness in a topological space.

2 Ultrafilter Compact Space

Let X be a fuzzy topological space. Let $S_X \subseteq F(X)$(=the class of all prefilters on X). Let $C_X \subseteq S_X \times P_X$ and $L_X \subseteq S_X \times P_X$.

(P1) $(\mathcal{F}, p) \in C_X$ if and only if there exists $\mathcal{G} \in S_X$ such that $\mathcal{G} \supseteq \mathcal{F}$ and $(\mathcal{G}, p) \in L_X$

(P2) for every element $\mathcal{F} \in S_X$, there exists a maximal element in S_X containing $\mathcal{F}$ with respect to inclusion.

(P3) if $f : X \to Y$ is a map and $\mathcal{F}$ is a maximal element in S_X, then $f(\mathcal{F})$ is a maximal element in S_Y.

(P4) $(\mathcal{F}, p) \in L_{\prod_\alpha X_\alpha}$ if and only if $(\pi_\alpha(\mathcal{F}), \pi_\alpha(p)) \in L_{X_\alpha}$ for each α.

From now on, we assume that every fuzzy topological space X is equipped with a triple (S_X, C_X, L_X) satisfying **(P1)**, **(P2)**, **(P3)** and **(P4)**.

Definition 2.1 A space X is called an *ultrafilter compact space* if for every maximal element $\mathcal{U}$ in S_X, there exists $p \in P(X)$ such that $(\mathcal{U}, p) \in L_X$.

Theorem 2.2 *A space X is ultrafilter compact if and only if for every element $\mathcal{F}$ in S_X there exists $p \in P(X)$ such that $(\mathcal{F}, p) \in C_X$.*

Theorem 2.3 **(Tychonoff)** *Let $\{X_\alpha\}_\Lambda$ be a family of spaces. Then the product space $\prod_\alpha X_\alpha$ is ultrafilter compact if and only if so is X_α for each α.*

3 Examples

We give several examples of ultrafilter fuzzy compact spaces by introducing triples (S_X, C_X, L_X) satisfying **(P1)**, **(P2)**, **(P3)** and **(P4)**.

A. In a fuzzy topological space, there exist many different types of neighborhoods of a (fuzzy) point. See [2,7,9].

A prefilter $\mathcal{F}$ on X converges to $x(p)$ if and only if $\mathcal{F} \supseteq \mathfrak{N}_x(\mathfrak{N}_p)$, where $\mathfrak{N}_x(\mathfrak{N}_p)$ is the neighborhood filter of $x(p)$, respectively.

A point $x(p)$ in X is a cluster point of a prefilter $\mathcal{F}$ on X if $F \cap U \neq \emptyset$ for all $F \in \mathcal{F}$ and $\mathcal{U} \in \mathfrak{N}_x(\mathfrak{N}_p)$, respectively.

Let $S_X = F(X)$, $C_X = \{(\mathcal{F}, p) \mid p$ is a cluster point of $\mathcal{F}$ in $X\}$ and $L_X = \{(\mathcal{F}, p) \mid \mathcal{F}$ converges to p in $X\}$.

B. Let S_X be the collection of all t-prefilters[8] on X, $C_X = \{(\mathcal{F}, p) \mid p$ is a t-cluster point of $\mathcal{F}$ in $X\}$ and $L_X = \{(\mathcal{F}, p) \mid \mathcal{F}$ t-converges to p in $X\}$. (See [8]). For this example, we use functors ι_α and ω_α between fuzzy topological spaces and topological spaces, where $\iota_\alpha((X, \delta)) = (X, \iota(\delta))$, $\iota_\alpha(\delta) = \{\mu^{-1}(\alpha, 1] \mid \mu \in \delta\}$ and $\omega_\alpha((X, \mathcal{T})) = (X, \omega_\alpha(\mathcal{T})), \omega_\alpha(\mathcal{T}) = \{\mu \in I^X \mid \mu^{-1}(t, 1] \in \mathcal{T}\}$. Moreover, using the functors j_α and υ_α introduced in [5] we can obtain another example.

C. Let S_X be the collection of all $\bar{\alpha}$-filters [6] on X $(0 < \alpha < 1)$, $C_X = \{(\mathcal{F}, p) \mid p$ is an α-cluster point of $\mathcal{F}$ in $X\}$ and $L_X = \{(\mathcal{F}, p) \mid \mathcal{F}$ converges to p in $X\}$. In this case, the notion of ultrafilter compactness is exactly same as that of stable fuzzy compactness [6].

Remark 1 By a similar method, we may extend this result for the case of different types of convergence, e.g. convergence to a fuzzy set[4].

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