Likely Mean Value Theorem

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I. Introduction

In this paper, we study the likely mean value theorem with respect to the integral of fuzzy mapping between fuzzy bounds. The ideas of integration of fuzzy mappings are investigated in [1], [2], [3] and [4].

II. Definitions and Main Results

A fuzzy domain $\mathcal{D}$ of the real line $\boldsymbol{R}$ is assumed to be delimited by two fuzzy bounds $\tilde{a}$ and $\tilde{b}$ in the following sense:

(1) $\tilde{a}$ and $\tilde{b}$ are fuzzy sets on $\boldsymbol{R}$, whose membership functions are $\mu_{\tilde{a}}$ and $\mu_{\tilde{b}}$ from $\boldsymbol{R}$ to $[0, 1]$.

(2) For all $x \in$ R, $\mu_{\tilde{a}}(x)$ (resp. $\mu_{\tilde{b}}(x)$) evaluates to what extent x can be considered as a greatest lower bound(resp. least upper bound) of $\mathcal{D}$.

(3) $\tilde{a}$ and $\tilde{b}$ are normalized, i. e., there exists

$$a, b \in \mathrm{R}, \qquad \text{such that} \qquad \mu_{\tilde{a}}(a) = 1 = \mu_{\tilde{b}}(b).$$

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(4) $\tilde{a}$ and $\tilde{b}$ are convex fuzzy sets, i.e., for any $\alpha \in (0, 1]$ their α–cut $\tilde{a}_\alpha$ and $\tilde{b}_\alpha$ are intervals.

$\mathcal{D}$ is denoted by $(\tilde{a}, \tilde{b})$: $\tilde{a}$ and $\tilde{b}$ are assumed to be ordered in the sense that

$$a_0 = \inf S(\tilde{a}) \leq \sup S(\tilde{b}) = b_0$$

where

$$S(\tilde{a}) = \{x | \mu_{\tilde{a}}(x) > 0\}$$

is the support of $\tilde{a}$ ([3]).

Definition 1. ([4]) A fuzzy number $\tilde{A}$ in the real line $\boldsymbol{R}$ is the fuzzy set characterized by a membership function $\mu_{\tilde{A}} : R \to [0, 1]$. A fuzzy number $\tilde{A}$ is expressed as $\tilde{A} = \int_{x \in R} \mu_{\tilde{A}}(x)/x$, with the understanding that $\mu_{\tilde{A}}(x) \in [0, 1]$ represents the grade of membership of x in $\tilde{A}$ and $\int$ denotes the union of $\mu_{\tilde{A}}(x)/x$'s.

Definition 2. ([4]) fuzzy number $\tilde{A}$ in $\boldsymbol{R}$ is said to be convex if for any real numbers $x, y, z \in \boldsymbol{R}$ with $x \leq y \leq z$, $\mu_A(y) \geq \mu_A(x) \wedge \mu_A(z)$, where $\wedge$ standing for minimum. A fuzzy number $\tilde{A}$ is called normal if the following $\max_x \mu_{\tilde{A}}(x) = 1$ holds. A fuzzy number which is normal and convex is referred to normal convex fuzzy number.

Remark. Let $\tilde{a}$ and $\tilde{b}$ be normal convex fuzzy number with the bounded support. $\tilde{a}$ and $\tilde{b}$ be assumed to be ordered in the sense that $a_0 = \inf S(\tilde{a}) \leq \sup S(\tilde{b}) = b_0$, where $S(\tilde{a}) = \{x | \mu_{\tilde{a}}(x) > 0\}$ is the support of $\tilde{a}$. Then $(\tilde{a}, \tilde{b})$ satisfies the definition of fuzzy domain.

Definition 3. ([5]) Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers. The membership function of their extended substraction $\tilde{A} \ominus \tilde{B}$ is defined by $\mu_{\tilde{A} \ominus \tilde{B}}(z) = \sup_{z=x-y} \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y))$.

Definition 4. ([2]) Let f be a real-valued real mapping, supposedly integrable on the interval $I = [\inf S(\tilde{A}), \sup S(\tilde{B})]$;then the integral of f over the domain delimited by the fuzzy bounds $\tilde{A}$ and $\tilde{B}$, denoted by $\int_D f$, is defined according to extension principle by

$$^{\forall}z \in \boldsymbol{R},\ \mu_{\int_D f}(z) = \sup_{x,y \in I} \min(\mu_{\tilde{a}}(x),\ \mu_{\tilde{b}}(y))$$

under the constraint $z = \int_x^y f$, where $\int_x^y f$ is short for $\int_x^y f(s)ds$.

$\int_D f$ will also be denoted by $\int_{\tilde{a}}^{\tilde{b}} f$.

Definition 5. ([3]) A fuzzy point $\tilde{c}$ is a convex subset of the real line $\boldsymbol{R}$ and its membership function is defined by

$$^{\forall}x,\ ^{\forall}y > x,\ ^{\forall}z \in [x,\ y],\ \mu_{\tilde{c}}(z) \geq \min(\mu_{\tilde{c}}(x),\ \mu_{\tilde{c}}(y)).$$

Theorem 1. ([7]) Let $\tilde{a}$ and $\tilde{b}$ be normal convex fuzzy numbers on $\boldsymbol{R}$ with the bounded support and f be a real-valued mapping, supposedly integrable on the interval $[\inf S(\tilde{a}),\ \sup S(\tilde{b})]$, then there exists a fuzzy point $\tilde{c}$ satisfying

$$\int_{\tilde{a}}^{\tilde{b}} f ds \subseteq f(\tilde{c})(\tilde{b} \ominus \tilde{a}),$$

where

$$S(\tilde{c}) \subset [\inf S(\tilde{a}),\ \sup S(\tilde{b})].$$

Definition 6.([2]) LR-fuzzy mapping is a fuzzy mapping $\tilde{f}$ such that $\tilde{f}(u)$ is an LR-type fuzzy number for all u, i. e. $\exists\, L,\ R : \boldsymbol{R}^+ \to [0,\ 1],\ \varphi : I \to \boldsymbol{R},\ a,\ b$ mappings from I to $\boldsymbol{R}^+$ such that:

$$^{\forall}u,\ \mu_{\tilde{a}(u)}(v) = \begin{cases} L\left(\dfrac{\varphi(u) - v}{a(u)}\right) & ^{\forall}v \leq \varphi(u), \\ \\ R\left(\dfrac{v - \varphi(u)}{b(u)}\right) & ^{\forall}v \geq \varphi(u), \end{cases}$$

where L and R be such that $L(0) = R(0) = 1,\ L(1) = R(1) = 0$ and decreasing, a and b positive mappings. L and R be called reference mappings, a and b spread mappings, φ is the strong mapping of $\tilde{f}$; it is a 1–curve. $\tilde{f}$ is also denoted

$$\tilde{f} = (\varphi,\ a,\ b)_{LR}$$

and

$$\tilde{f}(u) = (\varphi(u),\ a(u),\ b(u))_{LR}$$

Theorem 2. Let $\tilde{a}$ and $\tilde{b}$ be normal convex fuzzy numbers on $\boldsymbol{R}$ with the bounded support and $\tilde{f}$ be fuzzy mapping such that $\tilde{f}(\cdot) = M_1 f_1(\cdot) \oplus M_2 f_2(\cdot) \oplus \cdots \oplus M_n f_n(\cdot)$, where M_i is a LR-fuzzy number which is denoted by $M_i = (m_i,\ \underline{m}_i,\ \overline{m}_i)_{LR}$. If the real mapping f is integrable on the interval $[\inf S(\tilde{a}),\ \sup S(\tilde{b})]$, then there exists a fuzzy point $\tilde{c}$ satisfying

$$\int_{\tilde{a}}^{\tilde{b}} \tilde{f} \subseteq \tilde{f}(\tilde{c})(\tilde{b} \ominus \tilde{a}),$$

where $S(\tilde{c}) \subset [\inf S(\tilde{a}),\ \sup S(\tilde{b})]$.

PROOF. Since

$$\tilde{f}(\cdot) = M_1 f_1(\cdot) \oplus M_2 f_2(\cdot) \oplus \cdots \oplus M_n f_n(\cdot),$$

$$\int_{\tilde{a}}^{\tilde{b}} \tilde{f} = M_1 \int_{\tilde{a}}^{\tilde{b}} f_1 \oplus M_2 \int_{\tilde{a}}^{\tilde{b}} f_2 \oplus \cdots \oplus M_n \int_{\tilde{a}}^{\tilde{b}} f_n.$$

By Theorem 1, there exist $\tilde{c}_i$'s such that

$$\int_{\tilde{a}}^{\tilde{b}} f_i \subseteq f_i(\tilde{c}_i)(\tilde{b} \ominus \tilde{a}), \quad i = 1,\ 2,\ \cdots,\ n,$$

where $S(\tilde{c}_i) \subset [\inf S(\tilde{a}),\ \sup S(\tilde{b})]$. Thus we have

$$\begin{aligned}\int_{\tilde{a}}^{\tilde{b}} \tilde{f} &= M_1 \int_{\tilde{a}}^{\tilde{b}} f_1 \oplus M_2 \int_{\tilde{a}}^{\tilde{b}} f_2 \oplus \cdots \oplus M_n \int_{\tilde{a}}^{\tilde{b}} f_n \\ &\subseteq M_1 f_1(\tilde{c}_1)(\tilde{b} \ominus \tilde{a}) \oplus M_2 f_2(\tilde{c}_2)(\tilde{b} \ominus \tilde{a}) \oplus \cdots \oplus M_n f_n(\tilde{c}_n)(\tilde{b} \ominus \tilde{a}) \\ &= \tilde{f}(\tilde{c})(\tilde{b} \ominus \tilde{a}).\end{aligned}$$

Example 1. Let $\tilde{f}(x) = M_1 f_1(x) \oplus M_2 f_2(x) \oplus M_3 f_3(x)$, where $f_i(x) = 1$, for $i = 1, 2, 3$.

$$M_1 = (\varphi_1, a_1, b_1)_{LR} = \left(\frac{7}{3}, \frac{3}{8}, \frac{3}{4}\right)_{LR},$$
$$M_2 = (\varphi_2, a_2, b_2)_{LR} = \left(\frac{19}{3}, \frac{5}{8}, \frac{5}{4}\right)_{LR},$$
$$\text{and} \quad M_3 = (\varphi_3, a_3, b_3)_{LR} = \left(\frac{27}{3}, \frac{7}{8}, \frac{7}{4}\right)_{LR}.$$

Therefore,

$$\begin{aligned}\int_{\tilde{1}}^{\tilde{4}} \tilde{f} &= M_1 \int_{\tilde{1}}^{\tilde{4}} dx \oplus M_2 \int_{\tilde{1}}^{\tilde{4}} dx \oplus M_3 \int_{\tilde{1}}^{\tilde{4}} dx \\ &\subseteq (M_1\tilde{c} \oplus M_2\tilde{c} \oplus M_3\tilde{c})(\tilde{4} \ominus \tilde{1}) \qquad \text{(by Theorem 1)} \\ &= (M_1 \oplus M_2 \oplus M_3)\tilde{c}(\tilde{4} \ominus \tilde{1}) \\ &= M\tilde{c}(\tilde{4} \ominus \tilde{1}) \\ &= \tilde{f}(\tilde{c})(\tilde{4} \ominus \tilde{1}),\end{aligned}$$

where $M = (\varphi, a, b)_{LR} = (21, 1.875, 3.75)_{LR}$ and $S(\tilde{c}) \subset [\inf S(\tilde{1}), \sup S(\tilde{4})]$.

Theorem 3. If the fuzzy mappings $\tilde{f}$ and $\tilde{g}$ satisfy the conditions of theorem 2, then

$$\int_{\tilde{a}}^{\tilde{b}} (\tilde{f} \oplus \tilde{g}) \subseteq \int_{\tilde{a}}^{\tilde{b}} \tilde{f} \oplus \int_{\tilde{a}}^{\tilde{b}} \tilde{g}$$

Theorem 4. Let f and g be either positive or negative real mappings. If the fuzzy mappings $\tilde{f}$ and $\tilde{g}$ satisfy the conditions of theorem 2, then we have

$$\int_{\tilde{a}}^{\tilde{b}} (\tilde{f} + \tilde{g}) = \int_{\tilde{a}}^{\tilde{b}} \tilde{f} \oplus \int_{\tilde{a}}^{\tilde{b}} \tilde{g}$$

Theorem 5. Let D' and (D") be the domains of the real line $\boldsymbol{R}$ delimited by the fuzzy bounds $(\tilde{a}, \tilde{c})$ and $(\tilde{c}, \tilde{b})$, respectively. If $\tilde{f}$ and $\tilde{g}$ are fuzzy mappings, which satisfy the conditions of theorem 2, then

$$\int_{D} \tilde{f} \subseteq \int_{D'} \tilde{f} \oplus \int_{D''} \tilde{f},$$

where D is delimited by $(\tilde{a}, \tilde{b})$. The equality holds if and only if $\tilde{c}$ is a real number.

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