FUZZY r-SEMIOPEN SETS AND FUZZY r-SEMICONTINUOUS MAPS

EUN PYO LEE AND SEOK JONG LEE

Department of Mathematics, Seonam University, Namwon, Korea
Department of Mathematics, Chungbuk National University, Cheongju, Korea

INTRODUCTION

As a generalization of a set, the concept of fuzzy set was introduced by Zadeh. Chang[2] introduced fuzzy topological spaces and several other authors continued the investigation of such spaces. Some authors[4,5,6] introduced new definitions of fuzzy topology as a generalization of Chang's fuzzy topology. In this paper, we generalize the concepts of fuzzy semiopen and fuzzy semicontinuous of Azad[1]. We introduce the concepts of fuzzy r-semiopen(r-semiclosed) sets and fuzzy r-semicontinuous(r-open, r-closed) maps and then study some of their basic properties.

1. PRELIMINARIES

DEFINITION 1.1. [2] A *Chang's fuzzy topology* on X is a family T of fuzzy sets in X which satisfies the following properties :

(1) $\tilde{0}, \tilde{1} \in T$.
(2) If $\mu_1, \mu_2 \in T$ then $\mu_1 \wedge \mu_2 \in T$.
(3) If μ_i for each i, then $\bigvee \mu_i \in T$.

The pair (X, T) is called a *Chang's fuzzy topological space.*

DEFINITION 1.2. [4] A *fuzzy topology* on X is a map $\mathcal{T} : I^X \to I$ which satisfies the following properties :

(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$,
(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$,
(3) $\mathcal{T}(\bigvee \mu_i) \geq \bigwedge \mathcal{T}(\mu_i)$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space.*

DEFINITION 1.3. [3] Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy closure* is defined by

$$\mathrm{cl}(\mu, r) = \bigwedge\{\rho \in I^X | \mu \leq \rho, \mathcal{T}(\rho^c) \geq r\}.$$

From now on, for $r \in I_0$ we will call μ a *fuzzy r-open set* of X if $\mathcal{T}(\mu) \geq r$, μ a *fuzzy r-closed set* of X if $\mathcal{T}(\mu^c) \geq r$ and $\mathrm{cl}(\mu, r)$ the *fuzzy r-closure* of μ.

2. Fuzzy interior operator

Now, we are going to define fuzzy interior operator in $(X, \mathcal{T})$.

DEFINITION 2.1. Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy r-interior* is defined as follows :

$$\operatorname{int}(\mu, r) = \bigvee \{\rho \in I^X | \mu \geq \rho, \mathcal{T}(\rho) \geq r\}.$$

Obviously, $\operatorname{int}(\mu, r) = \mu$ for any r-open set μ. Moreover, we have the following results.

PROPOSITION 2.2. *Let $(X, \mathcal{T})$ be a fuzzy topological space and* $\operatorname{int} : I^X \times I_0 \to I^X$ *the fuzzy interior operator in $(X, \mathcal{T})$. Then for $\mu, \rho \in I^X$ and $r, s \in I_0$,*

(1) $\operatorname{int}(\tilde{0}, r) = \tilde{0}, \operatorname{int}(\tilde{1}, r) = \tilde{1}$.
(2) $\operatorname{int}(\mu, r) \leq \mu$.
(3) $\operatorname{int}(\mu, r) \geq \operatorname{int}(\mu, s)$ *if* $r \leq s$.
(4) $\operatorname{int}(\mu \wedge \rho, r) = \operatorname{int}(\mu, r) \wedge \operatorname{int}(\rho, r)$.
(5) $\operatorname{int}(\operatorname{int}(\mu, r), r) = \operatorname{int}(\mu, r)$.
(6) *If* $r = \bigvee \{s \in I_0 | \operatorname{int}(\mu, s) = \mu\}$, *then* $\operatorname{int}(\mu, r) = \mu$.

PROPOSITION 2.3. *Let* $\operatorname{int} : I^X \times I_0 \to I^X$ *be a map satisfying (1)-(4) of Proposition 2.2. Let $\mathcal{T} : I^X \to I$ be a map defined by*

$$\mathcal{T}(\mu) = \bigvee \{r \in I_0 | \operatorname{int}(\mu, r) = \mu\}.$$

Then $\mathcal{T}$ is a fuzzy topology on X such that

$$\operatorname{int} = \operatorname{int}_{\mathcal{T}}$$

iff (5) and (6) of Proposition 2.2 are satisfied by int.

If $\operatorname{int} : I^X \times I_0 \to I^X$ is a fuzzy interior operator on X, then for each $r \in I_0$, $\operatorname{int}_r : I^X \to I^X$ defined by

$$\operatorname{int}_r(\mu) = \operatorname{int}(\mu, r)$$

is a Chang's fuzzy interior on X.

Let $(X, \mathcal{T})$ be a fuzzy topological space. For an r-cut $\mathcal{T}_r = \{\mu \in I^X | \mathcal{T}(\mu) \geq r\}$, it is obvious that $(X, \mathcal{T}_r)$ is a Chang's fuzzy topological space for all $r \in I_0$.

PROPOSITION 2.4. *An operator* $\operatorname{int} : I^X \times I_0 \to I^X$ *is a fuzzy interior for the fuzzy topological space $(X, \mathcal{T})$ if and only if* $\operatorname{int}_r : I^X \to I^X$ *is a Chang's fuzzy interior for the Chang's fuzzy topological space $(X, \mathcal{T}_r)$ for all $r \in I_0$.*

3. Fuzzy r-semiopen sets

DEFINITION 3.1. Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is said to be

(1) *fuzzy r-semiopen* if there is a fuzzy r-open set ρ in X such that $\rho \leq \mu \leq \mathrm{cl}(\rho, r)$,

(2) *fuzzy r-semiclosed* if there is a fuzzy r-closed set ρ in X such that $\mathrm{int}(\rho, r) \leq \mu \leq \rho$.

THEOREM 3.2. *Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then the following statements are equivalent :*

(1) *μ is a fuzzy r-semiopen set.*

(2) *μ^c is a fuzzy r-semiclosed set.*

(3) $\mathrm{cl}(\mathrm{int}(\mu, r), r) \geq \mu$.

(4) $\mathrm{int}(\mathrm{cl}(\mu^c, r), r) \leq \mu^c$.

THEOREM 3.3. (1) *Any union of fuzzy r-semiopen sets is fuzzy r-semiopen.*

(2) *Any intersection of fuzzy r-semiclosed sets is fuzzy r-semiclosed.*

DEFINITION 3.4. Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy semi-closure* is defined by

$$\mathrm{scl}(\mu, r) = \bigwedge\{\rho \in I^X | \mu \leq \rho,\ \rho \text{ is fuzzy } r\text{-semiclosed}\}$$

and the *fuzzy semi-interior* is defined by

$$\mathrm{sint}(\mu, r) = \bigvee\{\rho \in I^X | \mu \geq \rho,\ \rho \text{ is fuzzy } r\text{-semiopen}\}.$$

Obviously $\mathrm{scl}(\mu, r)$ is the smallest fuzzy r-semiclosed set which contains μ and $\mathrm{sint}(\mu, r)$ is the greatest fuzzy r-semiopen set which contained in μ. Also we have $\mu \leq \mathrm{scl}(\mu, r) \leq \mathrm{cl}(\mu, r)$ and $\mu \geq \mathrm{sint}(\mu, r) \geq \mathrm{int}(\mu, r)$.

REMARK 3.5. It is obvious that every fuzzy r-open (r-closed) set is fuzzy r-semiopen (r-semiclosed). That the converse need not be true. It also shows that the intersection (union) of any two fuzzy r-semiopen (r-semiclosed) sets need not be fuzzy r-semiopen (r-semiclosed). Even the intersection (union) of a fuzzy r-semiopen (r-semiclosed) set with a fuzzy r-open (r-closed) set may fail to be fuzzy r-semiopen (r-semiclosed).

THEOREM 3.6. *Let μ be a fuzzy set of a fuzzy topological space $(X, \mathcal{T})$ and $r \in I_0$. Then μ is fuzzy r-semiopen (r-semiclosed) in $(X, \mathcal{T})$ if and only if μ is fuzzy semiopen (semiclosed) set in $(X, \mathcal{T}_r)$.*

Let (X, T) be a Chang's fuzzy topological space and $r \in I_0$. Recall that a fuzzy topology $T^r : I^X \to I$ is defined by $T^r(\mu) = 1$ if $\mu = \tilde{0}, \tilde{1}$, r if $\mu \in T - \{\tilde{0}, \tilde{1}\}$ and 0 otherwise.

THEOREM 3.7. *Let μ be a fuzzy set of a Chang's fuzzy topological space (X, T) and $r \in I_0$. Then μ is fuzzy semiopen (semiclosed) in (X, T) if and only if μ is fuzzy r-semiopen (r-semiclosed) in (X, T^r).*

4. Fuzzy r-semicontinuous maps

Definition 4.1. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is called

(1) a *fuzzy r-continuous* map if $f^{-1}(\mu)$ is a fuzzy r-open set of X for each fuzzy r-open set μ of Y,

(2) a *fuzzy r-open* map if $f(\mu)$ is a fuzzy r-open set of Y for each fuzzy r-open set μ of X,

(3) a *fuzzy r-closed* map if $f(\mu)$ is a fuzzy r-closed set of Y for each fuzzy r-closed set μ of X.

Definition 4.2. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is called

(1) a *fuzzy r-semicontinuous* map if $f^{-1}(\mu)$ is a fuzzy r-semiopen set of X for each fuzzy r-open set μ of Y,

(2) a *fuzzy r-semiopen* map if $f(\mu)$ is a fuzzy r-semiopen set of Y for each fuzzy r-open set μ of X,

(3) a *fuzzy r-semiclosed* map if $f(\mu)$ is a fuzzy r-semiclosed set of Y for each fuzzy r-closed set μ of X.

Remark 4.3. In view of Remark 3.5, a fuzzy r-continuous(r-open, r-closed) map is also a fuzzy r-semicontinuous(r-semiopen, r-semiclosed) map for each $r \in I_0$. That the converse need not be true.

Theorem 4.4. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map from a fuzzy topological space X to another fuzzy topological space Y and $r \in I_0$. Then f is fuzzy r-semicontinuous(r-semiopen, r-semiclosed) if and only if $f : (X, \mathcal{T}_r) \to (Y, \mathcal{U}_r)$ is fuzzy semicontinuous (semiopen, semiclosed).*

Theorem 4.5. *Let $f : (X, T) \to (Y, U)$ be a map from a Chang's fuzzy topological space X to another Chang's fuzzy topological space Y and $r \in I_0$. Then f is fuzzy semicontinuous (semiopen, semiclosed) if and only if $f : (X, T^r) \to (Y, U^r)$ is fuzzy r-semicontinuous (r-semiopen, r-semiclosed).*

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