국부구조화 방법을 이용한 6자유도 병렬형 매니퓰레이터의 특이점 해석

Singularity Analysis of 6-DOF Parallel Manipulator with Local Structurization Method

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Abstracts *Generally, singularity analysis of 6-DOF parallel manipulators is very difficult and, as a result, velocity relation has many uncertainties. In this paper, an alternative method using the local structurization method(LSM) for the analysis of singular configurations is presented. With LSM, the velocity relation can be represented in a simple form, and the result is totally equivalent to the conventional velocity relation. The velocity relation suggested in this paper gives a closed-form solution of singularities.*

Keywords *Singularities, Redundant Sensor, 6-DOF Parallel Manipulator*

1 Introduction

The singularity problem of parallel manipulators has been studied by many researchers. However, the complicated nature of parallel manipulators makes it difficult to analyze.

In 1978, Hunt[1] found a singular configuration where all the six leg vectors intersect on one line, and the moving plate can rotate about the line. In 1986, Fichter[2] found another singular configuration where the moving plate parallel to the base is rotated about the z-axis by $\pm 90^o$. In 1988, Merlet[4] used the Grassmann Geometry to find singular configurations. He found again Hunt's, and Fichter's singular configurations and much more configurations with the Grassmann Geometry. In 1990, Gosselin and Angeles[5] classified the singularities of closed-loop kinematic chain mechanism into three categories in general form. In 1991, Ma and Angeles[6] suggested another classification of singular configurations, and especially, they derived the conditions for architecture singularity. In 1993, Sefrioui and Gosselin[7] derived analytical expressions of the singularity loci of a planar 3-DOF parallel manipulator. They showed that the singularity loci are located outside of the workspace in some cases.

From the kinematic viewpoint, singular configurations are occurred when derivatives of the kinematic equations are ill-conditioned. However, the complexity of the Jacobian matrix makes it difficult to find the ill-condition ,*i.e.*, singularity condition. The kinematics of a parallel manipulator can be solved with extra sensors. However, most researchers didn't pay attention to the effect of the sensors on the velocity relation. In this paper, The velocity relation and singularity of a 6-6 Stewart platform-type parallel manipulator(SPM) is investigated with the sensor information. The method to solve the kinematic problem is based on LSM[8]. LSM can solve the kinematics of several kinds of SPMs. However, we only consider a 6-6 SPM in this paper. The notations and the 6-6 SPM are described in Fig. 1.

2 Position kinematics

Kinematic relations of parallel manipulators are expressed in loop equations. In Fig. 1, one of them are represented, and the equation is

$$\boldsymbol{q}_i = \boldsymbol{p} + \boldsymbol{p}_i - \boldsymbol{b}_i \quad \text{for} \quad i = 1, 2, \cdots, 6 \tag{1}$$

and $\boldsymbol{p}_i = \boldsymbol{R}\boldsymbol{p}'_i$. $\boldsymbol{R}$ is z-y-x Euler rotation matrix, and $\boldsymbol{p}'_i$ is a vector from O_p to point P_i with respect to the moving coordinate frame. Eq. (1) is the governing equation of parallel manipulators, and most relations are derived from this equation. The inverse kinematics is a trivial problem. If we know the orientation and position, then the right hand side of Eq. (1) is known value. Thus, to get the leg length q_i, merely take a scalar product of Eq. (1).

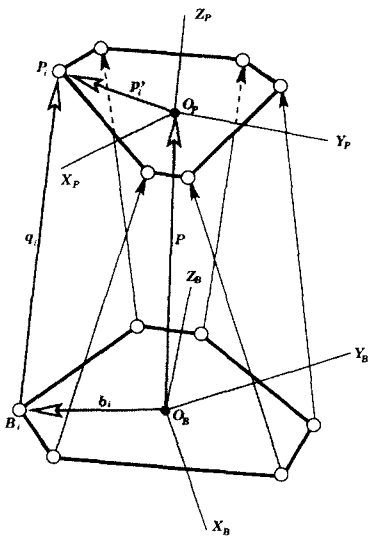

Fig. 1: Notations and kinematic constraints

2.1 Review on LSM for a 6-6 SPM

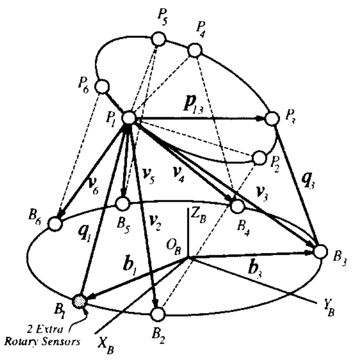

Fig. 2: G/P mechanism of a 6-6 SPM

LSM[8] uses G/P mechanism with 2 extra sensors as shown in Fig. 2. G/P mechanism means a mechanism which is composed of the globular and prismatic joint. G/P mechanism can be structurized with the virtual link or passive link. For a 6-6 SPM, the sensors are located at one of the base joints, for example, at point B_1 as shown in Fig. 2. With the sensors, the leg vector $\boldsymbol{q}_1$ can be fully defined. In this case, G/P mechanisms are defined as the triangle $\triangle P_1 B_i P_i$. The points P_1 and B_i are known, and thus, the lengths $\overline{P_1 B_i}$, $\overline{B_i P_i}$, and $\overline{P_1 P_i}$ are known. Let the vector directed from P_1 to B_i be $\boldsymbol{v}_i$, and P_1 to P_i be $\boldsymbol{p}_{1i}$ as shown in Fig. 2, then G/P mechanism can be solved as follows:

$$\boldsymbol{v}_i \cdot \boldsymbol{p}_{1i} = \frac{1}{2}(v_i^2 + p_{1i}^2 - q_i^2) \tag{2}$$

for $i = 2, 3, \cdots, 6$. If Eq. (2) is solved for all i instantaneously, the orientation and position can be obtained.

3 Velocity kinematics

3.1 General velocity kinematics

From Eq. (1), we can derive the velocity equation. Differentiate Eq. (1) with time, then

$$\begin{bmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_6 \end{bmatrix} \dot{\boldsymbol{q}} = \begin{bmatrix} \boldsymbol{q}_1^T & (\boldsymbol{p}_1 \times \boldsymbol{q}_1)^T \\ \boldsymbol{q}_2^T & (\boldsymbol{p}_2 \times \boldsymbol{q}_2)^T \\ \vdots & \vdots \\ \boldsymbol{q}_6^T & (\boldsymbol{p}_6 \times \boldsymbol{q}_6)^T \end{bmatrix} \dot{\boldsymbol{x}} \tag{3}$$

where $\dot{\boldsymbol{q}} = [\dot{q}_1 \cdots \dot{q}_6]^T$, and $\dot{\boldsymbol{x}} = [\dot{\boldsymbol{p}}^T \ \boldsymbol{w}^T]^T$. Eq. (3) represents the velocity relation of a SPM. For convenience, let Eq. (3) be

$$\boldsymbol{A}_o \dot{\boldsymbol{q}} = \boldsymbol{B}_o \dot{\boldsymbol{x}} \tag{4}$$

The singularity analysis for this general equation is very complex and difficult. Now, we will derive an alternative velocity relation to analyze the singularity of the above system by using LSM.

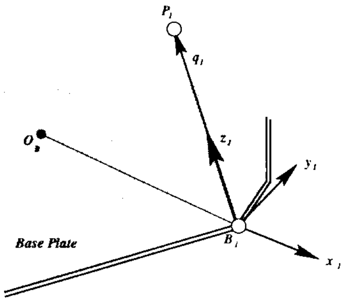

Fig. 3: Coordinate system of the leg vector $\boldsymbol{q}_1$

3.2 Velocity kinematics with LSM

Let us assume that the extra sensors are attached to the first leg, $\boldsymbol{q}_1$. Then the governing equation of LSM is Eq. (2). To obtain the velocity relation, differentiate Eq. (2), then

$$\dot{\boldsymbol{v}}_i \cdot \boldsymbol{p}_{1i} + \boldsymbol{v}_i \cdot \dot{\boldsymbol{p}}_{1i} = \boldsymbol{v}_i \cdot \dot{\boldsymbol{v}}_i + \boldsymbol{p}_{1i} \cdot \dot{\boldsymbol{p}}_{1i} - q_i \dot{q}_i \tag{5}$$

We have the following two kinematic constraints. First, for a rigid body, the velocity of two points along them are same, *i.e.*,

$$\dot{\boldsymbol{p}}_{1i} \cdot \boldsymbol{p}_{1i} = 0 \tag{6}$$

Second, two vector loop equations can be established to express the vector $\boldsymbol{v}_i$ as shown in Fig. 2,

$$\boldsymbol{v}_i = \boldsymbol{p}_{1i} - \boldsymbol{q}_i = -\boldsymbol{q}_1 + \boldsymbol{b}_{1i} \tag{7}$$

where $\boldsymbol{b}_{1i}$ is the vector from B_1 to B_i. In order to obtain the velocity form of $\boldsymbol{v}_i$, differentiate the above equation, then:

$$\dot{\boldsymbol{v}}_i = \dot{\boldsymbol{p}}_{1i} - \dot{\boldsymbol{q}}_i = -\dot{\boldsymbol{q}}_1 \tag{8}$$

With the relations of Eq. (6) $\sim$ (8), and $\dot{\boldsymbol{q}}_1 = \dot{\boldsymbol{p}} + \dot{\boldsymbol{p}}_1$, Eq. (5) can be arranged into the following two forms:

$$q_i \dot{q}_i - \boldsymbol{q}_i \cdot \dot{\boldsymbol{q}}_1 = -\boldsymbol{v}_i \cdot \dot{\boldsymbol{p}}_{1i} \tag{9}$$

$$q_i \dot{q}_i = \boldsymbol{q}_i \cdot \dot{\boldsymbol{p}} + \boldsymbol{q}_i \cdot \dot{\boldsymbol{p}}_i \tag{10}$$

With LSM, we can obtain additional values from the sensors attached to the base joint of leg 1. We can separate the sensor values from $\dot{\boldsymbol{q}}_1$ as follows:

$$\begin{bmatrix} q_1 & 0 & 0 & 0 & 0 & 0 \\ -\boldsymbol{q}_2^T \boldsymbol{a}_1 & q_2 & 0 & 0 & 0 & 0 \\ -\boldsymbol{q}_3^T \boldsymbol{a}_1 & 0 & q_3 & 0 & 0 & 0 \\ -\boldsymbol{q}_4^T \boldsymbol{a}_1 & 0 & 0 & q_4 & 0 & 0 \\ -\boldsymbol{q}_5^T \boldsymbol{a}_1 & 0 & 0 & 0 & q_5 & 0 \\ -\boldsymbol{q}_6^T \boldsymbol{a}_1 & 0 & 0 & 0 & 0 & q_6 \end{bmatrix} \dot{\boldsymbol{q}} = q_1 \begin{bmatrix} 0 & 0 \\ c\gamma_1 \boldsymbol{q}_2^T \boldsymbol{n}_1 & -\boldsymbol{q}_2^T \boldsymbol{s}_1 \\ c\gamma_1 \boldsymbol{q}_3^T \boldsymbol{n}_1 & -\boldsymbol{q}_3^T \boldsymbol{s}_1 \\ c\gamma_1 \boldsymbol{q}_4^T \boldsymbol{n}_1 & -\boldsymbol{q}_4^T \boldsymbol{s}_1 \\ c\gamma_1 \boldsymbol{q}_5^T \boldsymbol{n}_1 & -\boldsymbol{q}_5^T \boldsymbol{s}_1 \\ c\gamma_1 \boldsymbol{q}_6^T \boldsymbol{n}_1 & -\boldsymbol{q}_6^T \boldsymbol{s}_1 \end{bmatrix} \boldsymbol{\xi}$$

$$+ \begin{bmatrix} \boldsymbol{q}_1^T & (\boldsymbol{p}_1 \times \boldsymbol{q}_1)^T \\ \boldsymbol{0}^T & (\boldsymbol{p}_{12} \times \boldsymbol{q}_2)^T \\ \boldsymbol{0}^T & (\boldsymbol{p}_{13} \times \boldsymbol{q}_3)^T \\ \boldsymbol{0}^T & (\boldsymbol{p}_{14} \times \boldsymbol{q}_4)^T \\ \boldsymbol{0}^T & (\boldsymbol{p}_{15} \times \boldsymbol{q}_5)^T \\ \boldsymbol{0}^T & (\boldsymbol{p}_{16} \times \boldsymbol{q}_6)^T \end{bmatrix} \dot{\boldsymbol{x}} \tag{11}$$

where $\boldsymbol{n}_1$, $\boldsymbol{s}_1$, $\boldsymbol{a}_1$ and $\dot{\boldsymbol{\xi}}$ are expressed by the following relationships.

First, we define a coordinate system for the leg vector $\boldsymbol{q}_1$ as in Fig. 3. z_1-axis is in the direction of $\boldsymbol{q}_1$, x_1-axis is on the plane of $\overline{O_B B_1 P_1}$, and in the outward direction from O_B, and y_1-axis completes the right-hand coordinate system. The relation of the leg coordinate system and the base coordinate system is represented by the z-y-x Euler rotation matrix $\boldsymbol{R}_1 = [\boldsymbol{n}_1\ \boldsymbol{s}_1\ \boldsymbol{a}_1]$. α_1, β_1, and γ_1 are rotating angles of matrix $\boldsymbol{R}_1$ about z-, y-, and x-axis, respectively. Thus the position relation is

$$\boldsymbol{q}_1 = \boldsymbol{R}_1[0 \quad 0 \quad q_1]^T = q_1\boldsymbol{a}_1 \tag{12}$$

and the velocity relation is

$$\dot{\boldsymbol{q}}_1 = \dot{q}_1\boldsymbol{a}_1 + q_1\dot{\boldsymbol{a}}_1 \tag{13}$$

$\dot{\alpha}_1$ is always zero, *i.e.*, the leg coordinate system has always the same rotating angle about z-axis. α_1 is dependent on the geometry, and is determined as

$$\alpha_1 = \tan(B_{1y}/B_{1x}) \tag{14}$$

The result of $\dot{\boldsymbol{a}}_1$ is

$$\dot{\boldsymbol{a}}_1 = [c\gamma_1\boldsymbol{n}_1 \ -\boldsymbol{s}_1]\dot{\boldsymbol{\xi}} = \boldsymbol{L}\dot{\boldsymbol{\xi}} \tag{15}$$

where $\dot{\boldsymbol{\xi}} = [\dot{\beta}_1\ \dot{\gamma}_1]^T$.

By using Eq. (12) and Eq. (15), Eq. (11) can be obtained from Eq. (9) and Eq. (10).

With Eq. (13) and Eq. (15), $\dot{\boldsymbol{\xi}}$ can be obtained as follows:

$$\begin{aligned}\boldsymbol{q}_1 &= \boldsymbol{p} + \boldsymbol{p}_1 - \boldsymbol{b}_1 = \boldsymbol{a}_1 q_1\\ \dot{\boldsymbol{q}}_1 &= \dot{\boldsymbol{p}} + \dot{\boldsymbol{p}}_1 = \dot{\boldsymbol{a}}_1 q_1 + \boldsymbol{a}_1\dot{q}_1\\ \dot{\boldsymbol{a}}_1 q_1 &= q_1\boldsymbol{L}\dot{\boldsymbol{\xi}} = \dot{\boldsymbol{p}} + \dot{\boldsymbol{p}}_1 - \boldsymbol{a}_1\dot{q}_1\\ q_1\boldsymbol{L}^T\boldsymbol{L}\dot{\boldsymbol{\xi}} &= \boldsymbol{L}^T\dot{\boldsymbol{p}} + \boldsymbol{L}^T\dot{\boldsymbol{p}}_1 - \boldsymbol{L}^T\boldsymbol{a}_1\dot{q}_1\end{aligned} \tag{16}$$

$\boldsymbol{L}^T\boldsymbol{a}_1 = \boldsymbol{0}$, since $\boldsymbol{L}$ matrix is composed of $\boldsymbol{n}_1$ and $\boldsymbol{s}_1$ in the rotation matrix, which are orthogonal to $\boldsymbol{a}_1$. Thus, $\dot{\boldsymbol{\xi}}$ becomes

$$\begin{aligned}\dot{\boldsymbol{\xi}} &= \frac{1}{q_1}(\boldsymbol{L}^T\boldsymbol{L})^{-1}\{\boldsymbol{L}^T\dot{\boldsymbol{p}} + ([\boldsymbol{p}_1\times]\boldsymbol{L})^T\boldsymbol{w}\}\\ &= \frac{1}{q_1}\begin{bmatrix}\frac{1}{c\gamma_1}\boldsymbol{n}_1^T & \frac{1}{c\gamma_1}(\boldsymbol{p}_1\times\boldsymbol{n}_1)^T\\ -\boldsymbol{s}_1^T & -(\boldsymbol{p}_1\times\boldsymbol{s}_1)^T\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{p}}\\ \boldsymbol{w}\end{bmatrix}\\ &= \frac{1}{q_1}\boldsymbol{D}\dot{\boldsymbol{x}}\end{aligned} \tag{17}$$

where $[\boldsymbol{p}_i\times]$ is the matrix for cross-product.

For convenience, Eq. (11) is rewritten as

$$\boldsymbol{A}\dot{\boldsymbol{q}} = \boldsymbol{B}\dot{\boldsymbol{x}} + q_1\boldsymbol{C}\dot{\boldsymbol{\xi}} \tag{18}$$

Eq. (18) and Eq. (17) are combined in order to convert the sensor information into the input-output relation, then

$$\boldsymbol{A}\dot{\boldsymbol{q}} = \boldsymbol{B}\dot{\boldsymbol{x}} + q_1\boldsymbol{C}(\frac{1}{q_1}\boldsymbol{D}\dot{\boldsymbol{x}}) = (\boldsymbol{B} + \boldsymbol{C}\boldsymbol{D})\dot{\boldsymbol{x}} = \boldsymbol{B}'\dot{\boldsymbol{x}} \tag{19}$$

The result of $\boldsymbol{B}'$ matrix is shown in Appendix A.

After using Gauss-elimination procedure Eq. (19) can be derived as follows:

$$\boldsymbol{A}_n\dot{\boldsymbol{q}} = \boldsymbol{B}_n\dot{\boldsymbol{x}} \tag{20}$$

where $\boldsymbol{A}_n$ and $\boldsymbol{B}_n$ are represented in Appendix A.

4 Equivalence of Jacobian

Eq. (20) is much more simple to analyze the singularities of 6-DOF parallel manipulator when we compare the complexity of conventional Eq. (4).

In order to use Eq. (20), the equivalence of the Jacobian matrix with the original system shown in Eq. (4) should be considered. The equivalence of the two Jacobian matrices means that the characteristics, such as the singularity, are equivalent to each other.

The Jacobian matrix of the original system is defined as

$$\boldsymbol{J}_o = \boldsymbol{A}_o^{-1}\boldsymbol{B}_o = \begin{bmatrix}\hat{\boldsymbol{q}}_1^T & (\boldsymbol{p}_1\times\hat{\boldsymbol{q}}_1)^T\\ \vdots & \vdots\\ \hat{\boldsymbol{q}}_6^T & (\boldsymbol{p}_6\times\hat{\boldsymbol{q}}_6)^T\end{bmatrix} \tag{21}$$

where $\hat{\boldsymbol{q}}_i$ means the unit vector of $\boldsymbol{q}_i$.

Similarly, the Jacobian of the suggested system is defined as $\boldsymbol{J}_n = \boldsymbol{A}_n^{-1}\boldsymbol{B}_n$. Matrices $\boldsymbol{A}_n$ and $\boldsymbol{B}_n$ are derived from matrices $\boldsymbol{A}$ and $\boldsymbol{B}'$ by using Gauss-elimination with matrix $\boldsymbol{E}$. Gauss-elimination doesn't affect on the system property, thus the Jacobian $\boldsymbol{J}_n$ can be rewritten as the following equivalent form:

$$\boldsymbol{J}_n = \boldsymbol{A}_n^{-1}\boldsymbol{B}_n = (\boldsymbol{E}\boldsymbol{A})^{-1}(\boldsymbol{E}\boldsymbol{B}') = \boldsymbol{A}^{-1}\boldsymbol{B}' \tag{22}$$

where

$$\boldsymbol{A}^{-1} = \begin{bmatrix}1/q_1 & 0 & 0 & 0 & 0 & 0\\ \boldsymbol{q}_2^T\boldsymbol{a}_1/q_1q_2 & 1/q_2 & 0 & 0 & 0 & 0\\ \boldsymbol{q}_3^T\boldsymbol{a}_1/q_1q_3 & 0 & 1/q_3 & 0 & 0 & 0\\ \boldsymbol{q}_4^T\boldsymbol{a}_1/q_1q_4 & 0 & 0 & 1/q_4 & 0 & 0\\ \boldsymbol{q}_5^T\boldsymbol{a}_1/q_1q_5 & 0 & 0 & 0 & 1/q_5 & 0\\ \boldsymbol{q}_6^T\boldsymbol{a}_1/q_1q_6 & 0 & 0 & 0 & 0 & 1/q_6\end{bmatrix}$$

Since the first row of $\boldsymbol{J}_n$ is the same as the original one, and the remaining rows have the same form, one of the remaining rows are considered here for proof.

$$\begin{aligned}&\frac{\boldsymbol{q}_i^T\boldsymbol{a}_1}{q_1q_i}\boldsymbol{b}_{11}^T + \frac{1}{q_i}\boldsymbol{b}_{i1}^T\\ &= \frac{\boldsymbol{q}_i^T\boldsymbol{a}_1}{q_1q_i}\boldsymbol{q}_1^T + \frac{1}{q_i}(l_{i1}\boldsymbol{n}_1^T + l_{i2}\boldsymbol{s}_1^T)\\ &= \frac{\boldsymbol{q}_i^T}{q_i}(\frac{\boldsymbol{a}_1\boldsymbol{q}_1^T}{q_1} + \boldsymbol{n}_1\boldsymbol{n}_1^T + \boldsymbol{s}_1\boldsymbol{s}_1^T)\\ &= \hat{\boldsymbol{q}}_i^T(\frac{\boldsymbol{a}_1\boldsymbol{a}_1^T q_1}{q_1} + \boldsymbol{n}_1\boldsymbol{n}_1^T + \boldsymbol{s}_1\boldsymbol{s}_1^T)\\ &= \hat{\boldsymbol{q}}_i^T(\boldsymbol{n}_1\boldsymbol{n}_1^T + \boldsymbol{s}_1\boldsymbol{s}_1^T + \boldsymbol{a}_1\boldsymbol{a}_1^T)\\ &= \hat{\boldsymbol{q}}_i^T[\boldsymbol{n}_1\ \boldsymbol{s}_1\ \boldsymbol{a}_1][\boldsymbol{n}_1\ \boldsymbol{s}_1\ \boldsymbol{a}_1]^T\\ &= \hat{\boldsymbol{q}}_i^T\end{aligned} \tag{23}$$

for $i = 2, \cdots, 6$.

Similarly, the other three elements of a row are

$$\begin{aligned}
&\frac{\boldsymbol{q}_i^T \boldsymbol{a}_1}{q_1 q_i}\boldsymbol{b}_{12}^T + \frac{1}{q_i}\boldsymbol{b}_{i2}^T \\
&= \frac{\boldsymbol{q}_i^T \boldsymbol{a}_1}{q_1 q_i}(\boldsymbol{p}_1 \times \boldsymbol{q}_1)^T \\
&\quad + \frac{1}{q_i}\{(\boldsymbol{p}_{1i} \times \boldsymbol{q}_i)^T + l_{i1}(\boldsymbol{p}_1 \times \boldsymbol{n}_1)^T + l_{i2}(\boldsymbol{p}_1 \times \boldsymbol{s}_1)^T\} \\
&= \frac{\boldsymbol{q}_i^T \boldsymbol{a}_1}{q_1 q_i}(\boldsymbol{p}_1 \times \boldsymbol{q}_1)^T \\
&\quad + \frac{1}{q_i}\{(\boldsymbol{p}_i \times \boldsymbol{q}_i)^T - (\boldsymbol{p}_1 \times \boldsymbol{q}_i)^T \\
&\qquad + \boldsymbol{q}_i^T \boldsymbol{n}_1(\boldsymbol{p}_1 \times \boldsymbol{n}_1)^T + \boldsymbol{q}_i^T \boldsymbol{s}_1(\boldsymbol{p}_1 \times \boldsymbol{s}_1)^T\} \\
&= \frac{(\boldsymbol{p}_i \times \boldsymbol{q}_i)^T}{q_i} - \frac{(\boldsymbol{p}_1 \times \boldsymbol{q}_i)^T}{q_i} \\
&\quad + \frac{\boldsymbol{q}_i^T}{q_i}\{\frac{\boldsymbol{a}_1}{q_1}(\boldsymbol{p}_1 \times \boldsymbol{q}_1)^T + \boldsymbol{n}_1(\boldsymbol{p}_1 \times \boldsymbol{n}_1)^T + \boldsymbol{s}_1(\boldsymbol{p}_1 \times \boldsymbol{s}_1)^T\} \\
&= (\boldsymbol{p}_i \times \hat{\boldsymbol{q}}_i)^T - (\boldsymbol{p}_1 \times \hat{\boldsymbol{q}}_i)^T \\
&\quad + \hat{\boldsymbol{q}}_i^T\{\boldsymbol{a}_1([\boldsymbol{p}_1 \times]\boldsymbol{a}_1)^T + \boldsymbol{n}_1([\boldsymbol{p}_1 \times]\boldsymbol{n}_1)^T + \boldsymbol{s}_1([\boldsymbol{p}_1 \times]\boldsymbol{s}_1)^T\} \\
&= (\boldsymbol{p}_i \times \hat{\boldsymbol{q}}_i)^T - (\boldsymbol{p}_1 \times \hat{\boldsymbol{q}}_i)^T \\
&\quad + \hat{\boldsymbol{q}}_i^T\{\boldsymbol{a}_1\boldsymbol{a}_1^T[\boldsymbol{p}_1 \times]^T + \boldsymbol{n}_1\boldsymbol{n}_1^T[\boldsymbol{p}_1 \times]^T + \boldsymbol{s}_1\boldsymbol{s}_1^T[\boldsymbol{p}_1 \times]^T\} \\
&= (\boldsymbol{p}_i \times \hat{\boldsymbol{q}}_i)^T - (\boldsymbol{p}_1 \times \hat{\boldsymbol{q}}_i)^T \\
&\quad + \hat{\boldsymbol{q}}_i^T[\boldsymbol{n}_1\ \boldsymbol{s}_1\ \boldsymbol{a}_1][\boldsymbol{n}_1\ \boldsymbol{s}_1\ \boldsymbol{a}_1]^T[\boldsymbol{p}_1 \times]^T \\
&= (\boldsymbol{p}_i \times \hat{\boldsymbol{q}}_i)^T - (\boldsymbol{p}_1 \times \hat{\boldsymbol{q}}_i)^T + \hat{\boldsymbol{q}}_i^T[\boldsymbol{p}_1 \times]^T \\
&= (\boldsymbol{p}_i \times \hat{\boldsymbol{q}}_i)^T - (\boldsymbol{p}_1 \times \hat{\boldsymbol{q}}_i)^T + (\boldsymbol{p}_1 \times \hat{\boldsymbol{q}}_i)^T \\
&= (\boldsymbol{p}_i \times \hat{\boldsymbol{q}}_i)^T
\end{aligned} \tag{24}$$

Thus the Jacobian matrix $\boldsymbol{J}_n$ is equal to $\boldsymbol{J}_o$. Further, we can prove that the determinant of $\boldsymbol{B}'$ is equal to that of $\boldsymbol{B}_o$ easily, as follows:

$$\begin{aligned}
\det(\boldsymbol{J}_o) &= \det(\boldsymbol{J}_n) \\
\det(\boldsymbol{A}_o^{-1}\boldsymbol{B}_o) &= \det(\boldsymbol{A}^{-1}\boldsymbol{B}') \\
\det(\boldsymbol{A}_o^{-1})\det(\boldsymbol{B}_o) &= \det(\boldsymbol{A}^{-1})\det(\boldsymbol{B}')
\end{aligned}$$

Since $\boldsymbol{A}_o^{-1}$ is a diagonal matrix and $\boldsymbol{A}$ is a lower triangular matrix, the determinant of the two matrices is just multiplication of the diagonal elements. Consequently, the determinant is always equal to each other, since they have same diagonal elements. Finally, we can obtain

$$\begin{aligned}
\det(\boldsymbol{B}_o) &= \det(\boldsymbol{B}') = \det(\boldsymbol{E})\det(\boldsymbol{B}') = \det(\boldsymbol{E}\boldsymbol{B}') \\
&= \det(\boldsymbol{B}_n)
\end{aligned} \tag{25}$$

since $\det(\boldsymbol{E}) = 1$.

We can conclude that the two systems are totally equivalent, thus the singularity of one system is that of the other system. The whole singularity of the original system can be found from the suggested simple system with LSM.

5 Singularity analysis

5.1 Conditions for singularity

Let us consider matrices $\boldsymbol{A}_n$ and $\boldsymbol{B}_n$ for the analysis of singular configuration. Matrix $\boldsymbol{A}_n$ is related to the first kind of singularities[1]. Determinant of $\boldsymbol{A}_n$ is just multiplication of diagonal elements, *i.e.*,

$$\det(\boldsymbol{A}_n) = q_1 q_2 q_3 q_4 q_5 q_6 \tag{26}$$

If the elements q_i are zero, matrix $\boldsymbol{A}_n$ is singular. However, the elements q_i can not be zero in real situation, since the prismatic joints must have some lengths in any case. Thus, matrix $\boldsymbol{A}_n$ can not be singular except that some of the actuators are in their limit lengths. When actuators are in their limit length, *i.e.*, q_{min} or q_{max}, the manipulator loses one or more degrees-of-freedom.

The second kind of singularities is related to matrix $\boldsymbol{B}_n$. Matrix $\boldsymbol{B}_n$ is much more complicated than matrix $\boldsymbol{A}_n$, but is less complicated than the original matrix, $\boldsymbol{B}_o$, in Eq. (4). Determinant of matrix $\boldsymbol{B}_n$ can be written as follows:

$$\det(\boldsymbol{B}_n) = \det(\boldsymbol{B}_{n_1})\det(\boldsymbol{B}_{n_2}) \tag{27}$$

where

$$\boldsymbol{B}_{n_1} = [\boldsymbol{b}_{n_{11}}\ \boldsymbol{b}_{n_{21}}\ \boldsymbol{b}_{n_{31}}]^T, \boldsymbol{B}_{n_2} = [\boldsymbol{b}_{n_{42}}\ \boldsymbol{b}_{n_{52}}\ \boldsymbol{b}_{n_{62}}]^T$$

When one or both of the matrices $\boldsymbol{B}_{n_1}$ and $\boldsymbol{B}_{n_2}$ is singular, then the whole matrix $\boldsymbol{B}_n$ is singular. Matrices $\boldsymbol{B}_{n_1}$ and $\boldsymbol{B}_{n_2}$ are (3×3) square matrix, thus the calculation of their determinant is much simpler than the original matrix in Eq. (4).

The third kind of singularities is architecture singularity. This kind of singularities are also related to matrix $\boldsymbol{B}_n$, and thus, this kind of singularities can be found.

A general form of singularity conditions are also derived, but it is not shown in this paper since the equations are so much complicated. For the availability of the suggested relation, a simple numerical example is included in the next section.

6 Conclusion

In this paper, an alternative form of velocity equation for 6-6 SPM is derived. General approach has great limitation in the analysis of singularity and it was nearly impossible to use Eq. (4) for detecting the singular configuration because of the complex nature of the system. However, with the equation derived from LSM, it becomes easy to determine the whole singularity conditions. With this result, we can design a 6-6 SPM free from singularity. Although this is a promising result, we need more research in order to understand the meaning of the singularities derived from LSM.

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A APPENDIX

• Matrix $\boldsymbol{B}'$:

$$\boldsymbol{B}' = \begin{bmatrix} \boldsymbol{b}_{11} \; \boldsymbol{b}_{21} \cdots \boldsymbol{b}_{61} \\ \boldsymbol{b}_{12} \; \boldsymbol{b}_{22} \cdots \boldsymbol{b}_{62} \end{bmatrix}^T$$

where

$$\begin{aligned}
\boldsymbol{b}_{11} &= \boldsymbol{q}_1 \\
\boldsymbol{b}_{12} &= (\boldsymbol{p}_1 \times \boldsymbol{q}_1) \\
\boldsymbol{b}_{21} &= l_{21}\boldsymbol{n}_1 + l_{22}\boldsymbol{s}_1 \\
\boldsymbol{b}_{22} &= (\boldsymbol{p}_{12} \times \boldsymbol{q}_2) + l_{21}(\boldsymbol{p}_1 \times \boldsymbol{n}_1) + l_{22}(\boldsymbol{p}_1 \times \boldsymbol{s}_1) \\
\boldsymbol{b}_{61} &= l_{61}\boldsymbol{n}_1 + l_{62}\boldsymbol{s}_1 \\
\boldsymbol{b}_{62} &= (\boldsymbol{p}_{16} \times \boldsymbol{q}_6) + l_{61}(\boldsymbol{p}_1 \times \boldsymbol{n}_1) + l_{62}(\boldsymbol{p}_1 \times \boldsymbol{s}_1)
\end{aligned}$$

and where $l_{i1} = \boldsymbol{q}_i \cdot \boldsymbol{n}_1$ and $l_{i2} = \boldsymbol{q}_i \cdot \boldsymbol{s}_1$.

• Matrix $\boldsymbol{A}_n$:

$$\boldsymbol{A}_n = \begin{bmatrix} a_{n_{11}} & 0 & 0 & 0 & 0 & 0 \\ a_{n_{21}} & a_{n_{22}} & 0 & 0 & 0 & 0 \\ a_{n_{31}} & a_{n_{32}} & a_{n_{33}} & 0 & 0 & 0 \\ a_{n_{41}} & a_{n_{42}} & a_{n_{43}} & a_{n_{44}} & 0 & 0 \\ a_{n_{51}} & a_{n_{52}} & a_{n_{53}} & 0 & a_{n_{55}} & 0 \\ a_{n_{61}} & a_{n_{62}} & a_{n_{63}} & 0 & 0 & a_{n_{66}} \end{bmatrix}$$

where

$$\begin{aligned}
a_{n_{11}} &= q_1 \\
a_{n_{21}} &= -\boldsymbol{q}_2^T \boldsymbol{a}_1 \\
a_{n_{31}} &= -(\boldsymbol{q}_3 + e_{32}\boldsymbol{q}_2)^T \boldsymbol{a}_1 \\
a_{n_{41}} &= -(\boldsymbol{q}_4 + e_{43}\boldsymbol{q}_3 + e_{42}\boldsymbol{q}_2)^T \boldsymbol{a}_1 \\
a_{n_{51}} &= -(\boldsymbol{q}_5 + e_{53}\boldsymbol{q}_3 + e_{52}\boldsymbol{q}_2)^T \boldsymbol{a}_1 \\
a_{n_{61}} &= -(\boldsymbol{q}_6 + e_{63}\boldsymbol{q}_3 + e_{62}\boldsymbol{q}_2)^T \boldsymbol{a}_1
\end{aligned}$$

$$\begin{aligned}
a_{n_{22}} &= q_2 & a_{n_{32}} &= e_{32}q_2 & a_{n_{42}} &= e_{42}q_2 \\
a_{n_{52}} &= e_{52}q_2 & a_{n_{62}} &= e_{62}q_2 & a_{n_{33}} &= q_3 \\
a_{n_{43}} &= e_{43}q_3 & a_{n_{53}} &= e_{53}q_3 & a_{n_{63}} &= e_{63}q_3 \\
a_{n_{44}} &= q_4 & a_{n_{55}} &= q_5 & a_{n_{66}} &= q_6
\end{aligned}$$

$$\begin{aligned}
e_{32} &= -m_3 & e_{42} &= -m_4 + n_4 m_3 \\
e_{43} &= -n_4 & e_{52} &= -m_5 + n_5 m_3 \\
e_{53} &= -n_5 & e_{62} &= -m_6 + n_6 m_3 \\
e_{63} &= -n_6 & & \\
m_i &= \frac{l_{i1}}{l_{21}} & n_i &= \frac{l_{21}l_{i2} - l_{22}l_{i1}}{l_{21}l_{32} - l_{22}l_{31}}
\end{aligned}$$

• Matrix $\boldsymbol{B}_n$:

$$\boldsymbol{B}_n = \begin{bmatrix} \boldsymbol{b}_{n_{11}} & \boldsymbol{b}_{n_{21}} & \boldsymbol{b}_{n_{31}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{b}_{n_{12}} & \boldsymbol{b}_{n_{22}} & \boldsymbol{b}_{n_{32}} & \boldsymbol{b}_{n_{42}} & \boldsymbol{b}_{n_{52}} & \boldsymbol{b}_{n_{62}} \end{bmatrix}^T$$

where

$$\begin{aligned}
\boldsymbol{b}_{n_{11}} &= \boldsymbol{q}_1 \qquad \boldsymbol{b}_{n_{21}} = l_{21}\boldsymbol{n}_1 + l_{22}\boldsymbol{s}_1 \\
\boldsymbol{b}_{n_{31}} &= (l_{32} + e_{32}l_{22})\boldsymbol{s}_1 \\
\boldsymbol{b}_{n_{12}} &= (\boldsymbol{p}_1 \times \boldsymbol{q}_1) \\
\boldsymbol{b}_{n_{22}} &= (\boldsymbol{p}_{12} \times \boldsymbol{q}_2) + l_{21}(\boldsymbol{p}_1 \times \boldsymbol{n}_1) + l_{22}(\boldsymbol{p}_1 \times \boldsymbol{s}_1) \\
\boldsymbol{b}_{n_{32}} &= (\boldsymbol{p}_{13} \times \boldsymbol{q}_3) + e_{32}(\boldsymbol{p}_{12} \times \boldsymbol{q}_2) + (l_{32} + e_{32}l_{22})(\boldsymbol{p}_1 \times \boldsymbol{s}_1) \\
\boldsymbol{b}_{n_{42}} &= (\boldsymbol{p}_{14} \times \boldsymbol{q}_4) + e_{43}(\boldsymbol{p}_{13} \times \boldsymbol{q}_3) + e_{42}(\boldsymbol{p}_{12} \times \boldsymbol{q}_2) \\
\boldsymbol{b}_{n_{52}} &= (\boldsymbol{p}_{15} \times \boldsymbol{q}_5) + e_{53}(\boldsymbol{p}_{13} \times \boldsymbol{q}_3) + e_{52}(\boldsymbol{p}_{12} \times \boldsymbol{q}_2) \\
\boldsymbol{b}_{n_{62}} &= (\boldsymbol{p}_{16} \times \boldsymbol{q}_6) + e_{63}(\boldsymbol{p}_{13} \times \boldsymbol{q}_3) + e_{62}(\boldsymbol{p}_{12} \times \boldsymbol{q}_2)
\end{aligned}$$