EXTENSION OF FUZZY LIE SUBALGEBRAS AND FUZZY LIE IDEALS ON U(L)

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ABSTRACT. In this note we will discuss extension of fuzzy Lie subalgebra and fuzzy Lie ideals of a Lie algebra L on universal enveloping algebra U(L) of L and will study some relations among them.

1. Introduction

Definition 1. ([H]) A vector space L over a field F with an operation $[\ \ ] : L \times L \longrightarrow L$ called the *bracket* of x and y is called a *Lie algebra* over F if the following axioms are satisfied :

(L1) The bracket operation is bilinear,
(L2) $[xx] = 0$, for all $x \in L$,
(L3) $[x[yz]] + [y[zx]] + [z[xy]] = 0$, for all $x, y, z \in L$.

Axiom (L3) is called the *Jacobi identity.*

A subspace K of L is called a *Lie subalgebra* of L if $[xy] \in K$ for all $x, y \in K$ and a subspace I of L is called a *Lie ideal* of L if for all $x \in I$, $y \in L$ implies $[xy] \in I$.

Definition 2. ([K]) A fuzzy set μ of L is called a *fuzzy Lie subalgebra* of L if for all $\alpha \in F$, $x, y \in L$, the following are satisfied :

(i) $\mu(x + y) \geq \min(\mu(x), \mu(y))$,
(ii) $\mu(\alpha x) \geq \mu(x)$,
(iii) $\mu([xy]) \geq \min(\mu(x), \mu(y))$.

We call μ a *fuzzy Lie ideal* of L if the condition (iii) is replaced by $\mu([xy]) \geq \max(\mu(x), \mu(y))$.

Definition 3. ([L]) Let R is a ring. A fuzzy set γ of R is called a *fuzzy subring* of R if for all $x, y \in L$ the following are satisfied :

(i) $\gamma(x - y) \geq \min(\gamma(x), \gamma(y))$,
(ii) $\gamma(xy) \geq \min(\gamma(x), \gamma(y))$.

We call γ a *fuzzy ideal* of R if (ii) is replaced by $\gamma(xy) \geq \max(\gamma(x), \gamma(y))$.

Definition 4. ([H]) A *universal enveloping algebra* of L is a pair (U, i) where U is an associative algrbra with 1 over F, $i : L \longrightarrow U$ is a linear map satisfying, for all $x, y \in L$

$$i([xy]) = i(x)i(y) - i(y)i(x) \quad \cdots \quad (*)$$

and the following holds ; for any associative F−algrbra U with 1 and any linear map $j : L \longrightarrow U$ satisfying $(*)$, there exists a unique homomorphism of an algebra $\phi : U \longrightarrow U$(sending 1 to 1) such that $\phi \circ i = j$. We denote it by $U(L)$.

Lemma 5. ([K]) *A fuzzy set μ of L is a fuzzy Lie subalgebra [resp. fuzzy Lie ideals] of L if and only if the level subsets $\mu_t = \{x \in L : \mu(x) \geq t\}$, for $0 \leq t \leq \mu(0)$, are Lie subalgebras [resp. Lie ideals] of L.*

Lemma 6. *I is an ideal of $U(L)$ if and only if $I \cap L$ is a Lie ideal of L.*

Proof. ($\Rightarrow$) For all $x \in I \cap L$ and $y \in L$, since I is an ideal of $U(L)$, $xy - yx = [xy] \in I$. Hence $I \cap L$ is a Lie ideal of L.

($\Leftarrow$) For all $x, y \in I$, since $I \cap L$ is a Lie ideal of L, $[xy] = xy - yx \in I \cap L$. Hence $xy \in I$ and $yx \in I$. This means that I is a ideal of $U(L)$.

Proposion 7. *If a fuzzy set σ is a fuzzy subring [resp. fuzzy ideal] of $U(L)$, then $\sigma|_L$ is a fuzzy Lie subalgebra [resp. Lie ideal] of L.*

Proof. For all $x, y \in L$, $\alpha \in F$, we prove that the following :
(i) $\sigma|_L(x + y) = \sigma(x + y) \geq \min(\sigma(x), \sigma(y)) = \min(\sigma|_L(x), \sigma|_L(y))$.
(ii) $\sigma|_L(\alpha x) = \sigma(\alpha x) \geq \sigma(x) = \sigma|_L(x)$.
(iii)

$$\begin{aligned}
\sigma|_L([xy]) =& \sigma([xy]) = \sigma(xy - yx) \\
\geq& \min(\sigma(xy), \sigma(yx)) \\
\geq& \min(\min(\sigma(x), \sigma(y))) \\
\geq& \min(\sigma(x), \sigma(y)) \\
=& \min(\sigma|_L(x), \sigma|_L(y)),
\end{aligned}$$

and respectively,

$$\begin{aligned}
\sigma|_L([xy]) =& \sigma([xy]) = \sigma(xy - yx) \\
\geq& \min(\sigma(xy), \sigma(yx)) \\
\geq& \min(\max(\sigma(x), \sigma(y))) \\
\geq& \max(\sigma(x), \sigma(y)) \\
=& \max(\sigma|_L(x), \sigma|_L(y)).
\end{aligned}$$

Proposion 8. *Let σ be a fuzzy subring [resp. fuzzy ideal] of $U(L)$. Level subsets σ_t, $0 \leq t \leq \sigma(0)$, are a subring [resp. ideal] of $U(L)$ if and only if level subsets $\sigma_t \cap L$, $0 \leq t \leq \sigma(0)$, are Lie subalgebras [resp. Lie ideals] of L.*

Proof. By Lemma 5 and 6.

Remark. Let σ be a fuzzy subring [resp. fuzzy ideal] of $U(L)$ and μ be a fuzzy Lie subalgebra [resp. fuzzy Lie ideal] of L, respectively. Then we have the questions : whether the following equation holds or not ?,

$$\mu_t = (\sigma|L)_t \cap L$$

Another question is when the above equation holds ?

We can easely see that if the converse of Proposion 7 holds, then the above equations hold. Now, it is not known that the converse of Proposion 7 holds.

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