Fuzzy Convergence and Compactness

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In this paper, we introduce four types of compactness(i.e.**W-compact, M-compact, P-compact, Q-compact)** in a fuzzy topological space. It is shown that all of these notions of compactness are good extenions and allow Tychonoff theorems. Moreover we obtain relationships among these notions.

1. Neighborhoods and Convergences

Let X be a nonempty set, I^X be the set of all functions from X to I.

Definition 1.1. Let (X, δ) be a fuzzy topological space. A fuzzy set $A \in I^X$ is said to be a fuzzy W-neighborhood of an ordinary point $x \in X$ if there exists $\rho \in \delta$ with $\rho \leq A$ and $\rho(x) = A(x) > 0$. The collection of all W-neighborhoods of an fuzzy point in a fuzzy topology is denoted by $N_W(x)$.

Definition 1.2. Let (X, δ) be a fuzzy topological space. A fuzzy set $A \in I^X$ is said to be a fuzzy M-neighborhood of x_λ if there is some open F-set μ in X such that $x_\lambda \in \mu$ and $\mu \leq A$, The collection of all M-neighborhoods of an fuzzy point x_λ in a fuzzy topology δ is denoted by $N_M(x_\lambda)$.

Definition 1.3. Let x_λ be an F-point in X and (X, δ) an f.t.s. A fuzzy set $A \in I^X$ is said to be an fuzzy P-neighborhood of x_λ if there is some open fuzzy set μ in X such that $\lambda < \mu(x)$ and $\mu \leq A$, The collection of all P-neighborhoods of an fuzzy point x_λ in a fuzzy topology δ is denoted by $N_P(x_\lambda)$.

Definition 1.4. Let x_λ be an F-point in X and (X, δ) an f.t.s. A fuzzy set $A \in I^X$ is said to be an fuzzy Q-neighborhood of x_λ if there is some open fuzzy set in X such that $x_\lambda \langle \mu$ and $\mu \leq A$, The collection of all Q-neighborhoods of an fuzzy point x_λ in a fuzzy topology δ is denoted by $N_P(x_\lambda)$.

Remarks. A Q-neighborhood of a fuzzy point generally does not contain the point itself.

It is shown [3] that a fuzzy filter $\mathcal{F}$ in X is a fuzzy ultra filter if and only if $\mathcal{F}$ has the following property : If $\mu \in I^X$ is such that $\mu \wedge \rho \neq 0$ for each $\rho \in \mathcal{F}$, then $\mu \in \mathcal{F}$.

Lemma 1.5. Let X be a set of points, $\sqcup$ the family of all ultrafilters on X, and $\sqcup_F$ the family of all fuzzy ultra prefilters. We define two maps :

$$f: \sqcup \rightarrow \sqcup_F \quad \text{by} \quad f(A) = \{\mu \in I^X \mid \text{supp}(\mu) \in A \text{ for all } A \in \sqcup\},$$

and

$$g: \sqcup_F \rightarrow \sqcup \quad \text{by} \quad g(\beta) = \{\text{supp}(\rho) \mid \rho \in \mathcal{F} \text{ for all } \mathcal{F} \in \sqcup_F\}.$$

Then f and g are well defined.

2. Compactness

Definition 2.1. An f.t.s. X is said to be fuzzy ultra W-compact (briefly f.u.W. compact) if every ultra prefilter on X is W-convergent with respect to a ordinary point $x \in X$. See [5]

Definition 2.2. An f.t.s. X is said to be fuzzy ultra M-compact (briefly f.u.M. compact) if every ultra prefilter on X is M-convergent with respect to fuzzy point x_λ.

Definition 2.3. An f.t.s. X is said to be fuzzy ultra P-compact (briefly f.u.P. compact) if every ultra-prefilter on X is P-convergent with respect to fuzzy point x_λ.

Definition 2.4. An f.t.s. X is said to be fuzzy ultra Q-compact (briefly f.u.Q. compact) if every ultra prefilter on X is Q-convergent with respect to fuzzy point x_λ

Given a topology $\Im$ on X, the corresponding fuzzy topology $\omega(\Im)$(see [12]) is the family of all fuzzy sets μ in X which are lower semicontinuous on $(X, \Im)$; $(X, \omega(\Im))$ is $C(X, I_r)$, the set of all continuous functions from $(X, \Im)$ to I_r, the unit interval I equipped with the right topolgy $\Im_r = \{(\alpha, 1] \mid \alpha \in I\} \bigcup \{I\}$.

Theorem 2.5. Let $(X, \Im)$ be a topological space on X. Then

(1) $(X, \Im)$ is compact if and only if $(X, \omega(\Im))$ is an f.u.W. compact.

(2) $(X, \Im)$ is compact if and only if $(X, \omega(\Im))$ is an f.u.M. compact.

(3) $(X, \Im)$ is compact if and only if $(X, \omega(\Im))$ is an f.u.P. compact.

(4) $(X, \Im)$ is compact if and only if $(X, \omega(\Im))$ is an f.u.Q. compact.

Recall the definition of fuzzy continuous(briefly F-continuous) and equivalent relations [4]

Theorem 2.6. Let $(X, \Im)$ and (X, ρ) are topological spaces and let $f: (X, \Im) \rightarrow (Y, \rho)$ be a function. If f is F-continuous then the following are satified :

(1) If X is f.u.W. compact, then Y is f.u.W. compact.

(2) If X is f.u.M. compact, then Y is f.u.M. compact.

(3) If X is f.u.P. compact, then Y is f.u.P. compact.

(4) If X is f.u.Q. compact, then Y is f.u.Q. compact.

Theorem 2.7. Let $(X_i)_{i \in J}$ be a family of nonempty fuzzy topological spaces and let $X = \prod_{i \in J} X_i$ with the product fuzzy topology $\Im$. Then following are satisfied :

(1) X is F.U.W. compact if and only if each X_i is F.U.W. compact.

(2) X is F.U.M. compact if and only if each X_i is F.U.M. compact.

(3) X is F.U.P. compact if and only if each X_i is F.U.P. compact.

(4) X is F.U.Q. compact if and only if each X_i is F.U.Q. compact.

Putting together the foregoing arguments this shows that we have the following :

$$\begin{array}{ccccc} & & \text{f. u. W} & & \\ & \swarrow & \downarrow & & \\ \text{f. u. M} & \rightarrow & \text{f. u. P} & \not\Leftrightarrow & \text{f. u. Q} \end{array}$$

Theorem. Let $(X, \mathfrak{T})$ be an f.t.s. Then

(1) If $(X, \mathfrak{T})$ is an f.u.W-compact, then it is an f.u.M-compact.

(2) If $(X, \mathfrak{T})$ is an f.u.M-compact, then it is an f.u.P-compact.

References

[1] L. A. Zadeh, Fuzzy Sets, Inform. and Contr 8, 338-353 (1965).

[2] C. L. Chang, Fuzzy Topological Spaces, J. of Math. Analysis and Applications 24, 182-190 (1968).

[3] M. A. DE Prada Vicente and M. Saralegui Aranguren, Fuzzy Filters, J. of Math. Analysis and Applications vol. 129, No. 2, February 1, 560-568 (1988).

[4] Pu Pao-Ming and Liu Ying-Ming, Fuzzy Topology.II. Product and Quotient Spaces, J. of Math. Analysis and Applications 77, 20-37 (1980).

[5] A. K. Katsaras, Fuzzy of fuzzy filters in fuzzy topological spaces Bull. Math. de la soc. sci. Math. de la R.S. de Roumanie Tome 27 (75), No. 2, 131-137 (1983).

[6] C. K. Wong, Fuzzy points and Local Properties of Fuzzy Topology, J. of Math. Analysis and Applications 46, 316-328 (1974).

[7] Pu Pao-Ming and Liu Ying-Ming, Fuzzy Topology. I. Neighborhood Structure of a Fuzzy Point and Moore-Smith Convergence, J. of Math. Analysis and Applications 76, 571-599 (1980).

[8] Rekha Srivastava, and S. N. Lac, and Arun K. Srivastava, Fuzzy Hausdorff Topological Spaces, J. of Math. Analysis and Applications 81, 497-506 (1981).

[9] R. H. Warren, Neighborhoods, Bases and Continuity in Fuzzy topological spaces, Rocky Mountain J. of math. Vol. 8, No. 3, 459-470 (1978).

[10] Hu Cheng-Ming, Fuzzy Topological Spaces, J. of Math. Analysis and Applications 110, 141-178 (1985).

[11] C. De Mitri and E. Pascali, Characterization of Fuzzy Topologies from Neighborhoods of Fuzzy Points, J. of Math. Analysis and Applications 93, 1-14 (1983).

[12] R. Lowen, Fuzzy Topological Spaces and Fuzzy Compactness, J. of Math.Analysis and Applications 56, 621-633 (1976).

[13] T. E. Gantner and R. C. Steinlager and R. H. Warren, Compactness in Fuzzy Topological spaces, J. of Math. Analysis and Applications 62, 547-562 (1978).

[14] H. Warren, Fuzzy Topologies Characterized by Neighborhood Systems, Rocky Mountain J. of Math. Vol. 9, No. 4, 761-764 (1979).

[15] A. K. Katsaras, On Fuzzy Proximity Spaces, J. of Math. Analysis and Applications 75, 571-583 (1980).

[16] R. Lowen, A Comparision of Different Compactness Notions in Fuzzy Topological Spaces, J. of Math. Analysis and Applications 64, 446-454 (1978).

[17] R. Lowen, Initial and Final Fuzzy Topologies and the Fuzzy Tychonoff Theorem, J. of Math. Analysis and Applications 58, 11-21 (1977).

[18] C. K. Wong, Fuzzy Topology : Product and Quotient Theorem, J. of Math. Analysis and Applications 45, 512-521 (1974).

[19] J. A. Goguen, The Fuzzy Tychonoff Theorem, J. of Math. Analysis and Applications 43, 734-742 (1973).

[20] R. Lowen, Fuzzy Neighborhood Spaces, Fuzzy Sets and Systems 7, 165-189 (1982).