Some Properties of the Usual Fuzzy Topological Spaces

Jeong-Yeol Choi and Ju-Ran Moon
Department of Mathematics
Wonkwang University

1. Introduction.

In this paper, we shall investigate some properties of the usual fuzzy topological space which are similar to those of the usual topological space and shall investigate some properties which hold in the usual topological space but do not hold in the usual fuzzy topological space.

2. Preliminaries.

In the sequel, closed unit interval [0, 1] in the real line will be denoted by I, while $I^0 = (0,1), I_0 = (0,1]$, $\mathbf{R} = (-\infty,\infty)$, $\mathbf{R}^+ = [0,\infty)$, and $\mathbf{R}_0^+ = (0,\infty)$.

The symbol ϕ will be used to denote the empty (fuzzy) set. We will denote the set of all fuzzy sets in X and the set of all membership functions of fuzzy sets in X by $\widetilde{P}(X)$ and I^X, respectively. The set $\widetilde{P}(X)$ is called the *fuzzy power set* of X.

Define a function S from $\widetilde{P}(X)$ to $P(X)$ by

$$S(A) = \{x \in X \mid \mu_A(x) > 0\}$$

for each $A \in \widetilde{P}(X)$. This ordinary subset $S(A)$ of X is called the ***support*** of a fuzzy set A.

A *fuzzy point* or a *fuzzy singleton* in a set X is a fuzzy set in X such that the degree of membership is zero everywhere except only one point, say x, where it takes a degree of membership, say α, in I_0. The fuzzy point is denoted by $p(x,\alpha)$, or shortly x_α and x and α are called its ***support*** and *value*, respectively, that is, x_α is called a ***fuzzy point with the support*** x ***and the value*** α. When $\alpha = 1$, we simply write x instead of x_1, that is x_1 denotes an ordinary point x. The set of all fuzzy points in X will be denoted by $F_p(X)$.

A fuzzy point x_α with $\alpha \in I^0$ is said to ***belong to*** a fuzzy set A or x_α is an ***element*** of A, denoted by $x_\alpha \lessdot A$, if and only if $\alpha < \mu_A(x)$. Of course, $x \lessdot A$ if and only if $\mu_A(x) = 1$.

Definition 2.1. [1] *The* ***usual fuzzy metric*** d *is a function from* $F_p(\mathbf{R}) \times F_p(\mathbf{R})$ *to* $F_p(\mathbf{R}^+)$ *defined by for all* $(x_\alpha, y_\beta) \in F_p(\mathbf{R}) \times F_p(\mathbf{R})$

$$d(x_\alpha, y_\beta) = |x - y|_{\alpha\wedge\beta}.$$

The pair $(\mathbf{R}, d)$ *is called the* ***usual fuzzy metric space*** *or* ***usual fms*** *for short.*

Definition 2.2. *A fuzzy set A in $\mathbf{R}$ with*

$$S(A) = (a, b)$$

$$\mu_A(x) = \begin{cases} \alpha & \text{if } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

*is called the **fuzzy open interval with the support** (a, b) **and the vlaue** α or simply **fuzzy open interval**. In this case, we denote A by*

$$(a, b)_\alpha.$$

3. Some properties of the Usual Fts.

Definition 3.1. *Let A be a fuzzy set in $\mathbf{R}$. A fuzzy point x_α is an **interior fuzzy point** of A if and only if for all λ with $0 < \lambda < \alpha$ there exists a positive real number ε such that*

$$(x - \varepsilon, x + \varepsilon)_\lambda \subset A.$$

*The set of all interior fuzzy points of A is called the **interior** of A and denoted by $\mathring{A}$ or* $\text{int}(A)$. *The fuzzy set A is said to be **fuzzy open** or simply **open** if and only if*

$$A = \bigcup_{x_\alpha \in \text{int}(A)} x_\alpha.$$

In particular, the (fuzzy) empty set ϕ is fuzzy open because $\text{int}(\phi) = \phi$.

The following is obvious by the definition of a fuzzy open set.

Lemma 3.2. *A fuzzy set A in $\mathbf{R}$ is fuzzy open if and only if for each $x \in S(A)$,*

$$x_{\mu(A)} \in \text{int}(A).$$

Proposition 3.3. *Every fuzzy open interval $(a, b)_\alpha$ is fuzzy open.*

Proposition 3.4. *A fuzzy set A in $\mathbf{R}$ is fuzzy open if and only if it is the union of fuzzy open intervals.*

From the proposition 3.4, we have the alternative definition of a fuzzy open set. In the sequel, we will use conveniently the following definition for a fuzzy open set.

Definition 3.5. *The union of an arbitrary family of fuzzy open intervals is called a **fuzzy open set** or simply **open set**. The family of all fuzzy open sets in $\mathbf{R}$ is called the **usual fuzzy topology** for $\mathbf{R}$ and denoted by $\mathcal{U}$. The pair $(\mathbf{R}, \mathcal{U})$ is called the **usual fuzzy topological space**, or **usual fts** for short.*

Proposition 3.6. *The usual fuzzy topology $\mathcal{U}$ has the following properties:*

(OS1) For each $\alpha \in I$, $(-\infty, \infty)_\alpha \in \mathcal{U}$, where $(-\infty, \infty)_0 = \phi$.

(OS2) If $\{A_j \in \widetilde{P}(\mathbf{R}) | \, j \in J\} \subseteq \mathcal{U}$, then $\bigcup_{j \in J} A_j \in \mathcal{U}$.

(OS3) If $A \in \mathcal{U}$ and $B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.

Remark 3.7. *Recall that every open subset of the real line* **R** *is the union of countable collection of disjoint open intervals. But this property does not generally hold for a fuzzy open set in the usual fts* $(\mathbf{R}, \mathcal{U})$.

For example, consider

$$A = (-\infty, 0) \cup (-1, 1)_{\frac{1}{2}} \cup (0, \infty).$$

Since A is the union of fuzzy open intervals, it is fuzzy open.

Suppose that A is the union of countable collection of disjoint fuzzy open intervals, say

$$A = \bigcup_{j \in J} (a_j, b_j)_{\alpha_j},$$

where J is countable and $(a_j, b_j)_{\alpha_j} \cap (a_k, b_k)_{\alpha_k} = \phi$ if $j \neq k$.

Since $0 \in S(A)$, there exists only one $j \in J$ such that $a_j < 0 < b_j$ and $\alpha_j = \frac{1}{2}$. Since $a_j \in S(A)$, there exists only one $k \in J$ such that $a_k < a_j < b_k$ and $\alpha_k = 1$. Therefore, we see that

$$(a_j, b_j)_{\frac{1}{2}} \cap (a_k, b_k) \neq \phi.$$

Hence this fuzzy open set A cannot be the union of countable disjoint fuzzy open intervals.

Proposition 3.8. *Every family of disjoint nonempty fuzzy open sets in* **R** *is countable.*

Remark 3.9. *Recall that the Lindelöf theorem: Every family $\mathcal{F}$ of open sets of the real line* **R** *contains a countable subfamily $\mathcal{C}$ with*

$$\bigcup_{U \in \mathcal{F}} U = \bigcup_{U \in \mathcal{C}} U.$$

But this property does not generally hold in the usual fts.

Consider the family of fuzzy open intervals.

$$\mathcal{F} = \{(\alpha, 1)_\alpha | \, \alpha \in I^0\},$$

and set

$$A = \bigcup_{\alpha \in I^0} (\alpha, 1)_\alpha.$$

Then, we see that the membership function μ_A of A is as follows:

$$\mu_A(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $< \alpha_n >$ be an arbitrary increasing sequence in I^0. Set

$$B = \bigcup_{n \in \mathbf{N}} (\alpha_n, 1)_{\alpha_n},$$

where $\mathbf{N}$ denotes the set of all national numbers. Then, we have that $B \subset A$ but $B \neq A$. For instance, we have that for some $n \in \mathbf{N}$

$$\mu_B\left(\frac{\alpha_n + \alpha_{n+1}}{2}\right) = \alpha_n < \frac{\alpha_n + \alpha_{n+1}}{2} = \mu_A\left(\frac{\alpha_n + \alpha_{n+1}}{2}\right).$$

Definition 3.10. *A fuzzy set A in $\mathbf{R}$ is said to be fuzzy closed or simply closed if and only if its complement A^c is fuzzy open.*

Proposition 3.11. *The collection of all fuzzy closed sets in $\mathbf{R}$ has the following properties:*

(CS1) For each $\alpha \in I$, $\mathbf{R}_\alpha$ is fuzzy closed.

(CS2) The intersection of an arbitrary family of fuzzy closed sets is fuzzy closed.

(CS3) The union of two (and hence of any finite number of) fuzzy closed sets is fuzzy closed.

Remark 3.12. *According to the propositions 3.6 and 3.11, for all $\alpha \in I$ the fuzzy sets $\mathbf{R}_\alpha$ are both fuzzy open and fuzzy closed. It is natural to ask whether or not there exists a fuzzy open and closed set other than these. The answer is affirmative. In fact, we have the following example.*

Let a fuzzy set A in $\mathbf{R}$ be characterized by

$$\mu_A(x) = \begin{cases} \frac{1}{3} & \text{if } x \leq 0, \\ x + \frac{1}{3} & \text{if } 0 \leq x \leq \frac{1}{3}, \\ \frac{2}{3} & \text{if } x \geq \frac{1}{3}. \end{cases}$$

Then, the complement A^c of A is characterized by

$$\mu_{A^c}(x) = \begin{cases} \frac{2}{3} & \text{if } x \leq 0, \\ -x + \frac{2}{3} & \text{if } 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{3} & \text{if } x \geq \frac{2}{3}. \end{cases}$$

Note that the fuzzy sets A and A^c are the unions of fuzzy open intervals as follows:

$$A = (-\infty, 1)_{\frac{1}{3}} \bigcup \Big[\bigcup_{\frac{1}{3} < x < \frac{2}{3}} (x - \frac{1}{3}, 1)_x \Big] \bigcup (\frac{1}{3}, \infty)_{\frac{2}{3}},$$

$$A^c = (-\infty, 1)_{\frac{2}{3}} \bigcup \Big[\bigcup_{\frac{1}{3} < x < \frac{2}{3}} (-1, \frac{2}{3} - x)_x \Big] \bigcup (0, \infty)_{\frac{1}{3}}.$$

Thus this fuzzy set A is both fuzzy open and closed.

In this case, the support $S(A)$ of A is the real line $\mathbf{R}$. *Of course, there exists a fuzzy set which is not both fuzzy open and closed though its support is the real line* $\mathbf{R}$. *For instance, the fuzzy set $B = (-\infty, 1)_{\frac{1}{2}} \cup (0, \infty)_{\frac{2}{3}}$ is fuzzy open but is not fuzzy closed though $S(B) = \mathbf{R}$.*

Recall that a nonempty proper subset of $\mathbf{R}$ *cannot be both open and closed in the usual topology for* $\mathbf{R}$. *We also have the similar property to this.*

Lemma 3.13. *In the usual fts $(\mathbf{R}, \mathcal{U})$, we have the followings:*

(1) If U is fuzzy open, then $S(U)$ is open.

(2) If F is fuzzy closed, then $S(F)$ is closed.

Proposition 3.14. *If the support $S(A)$ of a fuzzy set A in* $\mathbf{R}$ *is a nonempty proper subset of* $\mathbf{R}$, *then A cannot be both fuzzy open and closed in the usual fts $(\mathbf{R}, \mathcal{U})$.*

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