On Fuzzy Complete Continuous .

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1.Introduction.

We introduce the concept of fuzzy complete continuity and investigate some properties related to the fuzzy complete continuity.

2. Preliminiaries.

In this sectin, we will introduce some elementary concepts and results in fuzzy set theory. Also we will introduce some elementary concepts and results in fuzzy topological spaces. Throughout this paper, we will denote the unit interval [0,1] of the real line as I. X, Y and Z, etc, will denote sets. For a X, I^X denotes the collection of all the fuzzy sets in X. And $\underline{0}_X$ (simply, $\underline{0}$) and $\underline{1}_X$ (simply, $\underline{1}$) denote the empty fuzzy set and the hole fuzzy set in X given by $\underline{0}_X(x) = 0$ for all $x \in X$ and $\underline{1}_X(x) = 1$ for all $x \in X$, respectively.

Definition 2.1[3]. *Let X be a non - empty set. Then A is called a* **fuzzy set** *in X if $A \in I^X$.*

Definition 2.2[3]. *Let $f : X \to Y$ be a mapping and let $A \in I^X$ and $B \in I^Y$.*

(1) *The* **inverse image** *of B under f, denoted by $f^{-1}(B)$, is a fuzzy set in X defined by for all $x \in X$,*

$$[f^{-1}(B)](x) = B(f(x)) = (B \circ f)(x).$$

(2) *The* **image** *of A under f, denoted by $f(A)$, is a fuzzy set in Y defined by for each $y \in Y$,*

$$[f(A)](y) = \begin{cases} Sup_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

where $f^{-1}(y) = \{x \in X : f(x) = y\}$.

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Lemma 2.A[3]. *Let $f : X \to Y$ be a mapping. Then:*

(1) $f^{-1}(B^c) = [f^{-1}(B)]^c$ *for each* $B \in I^Y$.
(2) $f(A^c) \supset [f(A)]^c$ *for each* $A \in I^X$.
(3) *If* $B_1 \subset B_2$, *then* $f^{-1}(B_1) \subset f^{-1}(B_2)$, *where* $B_1, B_2 \in I^Y$.
(4) *If* $A_1 \subset A_2$, *then* $f(A_1) \subset f(A_2)$, *where* $A_1, A_2 \in I^X$.
(5) $B \supset f(f^{-1}(B))$ *for each* $B \in I^Y$.
(6) $A \subset f^{-1}(f(A))$ *for each* $A \in I^X$.
(7) *Let* $g : Y \to Z$ *be any mapping. Then* $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$ *for each* $C \in I^Z$.

We can easily obtain the following results :

Lemma 2.B [1]. *Let $f : X \to Y$ be mapping.*

(1) *If f is injective, then* $f^{-1}(f(A)) = A$ *for each* $A \in I^X$.
(2) *If f is surjective, then* $f(f^{-1}(B)) = B$ *for each* $B \in I^Y$.
(3) *If f is bijective, then* $[f(A)]^c = f(A^c)$.

Definition 2.3 [3,5,7]. *Let X be a non-empty set. Then a subset $\mathcal{T}$ of I^X is called a* **fuzzy topology** *on X if $\mathcal{T}$ satisfies the following conditions:*

(1) $\underline{0}_X, \underline{1}_X \in \mathcal{T}$.
(2) *If* $A, B \in \mathcal{T}$, *then* $A \cap B \in \mathcal{T}$.
(3) *If* $\{A_\alpha : \alpha \in \Lambda\}$ *is a subset of $\mathcal{T}$. then* $\cup_{\alpha \in \Lambda} A_\alpha \in \mathcal{T}$, *where Λ is an index set.*

Each member of $\mathcal{T}$ is called a $\mathcal{T}$ **- open** (simply, an **open**) fuzzy set in X. F is called a $\mathcal{T}$ **-closed** (simply, a **closed**) fuzzy set in X, if $F^c \in \mathcal{T}$. The pair $(X, \mathcal{T})$ is called a **fuzzy topological space** (simply, **fts**).

For a fts X, we will denote the collection of all the open (resp. closed) fuzzy sets in X as $OF(X)$ (resp.$CF(X)$).

For a set X, the collection $\{\underline{0}_X, \underline{1}_X\}$ is called the indiscrete fuzzy topology on X, and I^X the discrete fuzzy topology on X.

Definition 2.4 [5]. *Let X be a fts and let $A \in I^X$. Then :*

(1) *The union of all the open fuzzy sets in A is called the* **interior** *of A, and denote by $\mathring{A}$, $int_X A$ or $int A$. Hence,*

$$\mathring{A} = \cup\{O \in OF(X) : O \subset A\}.$$

(2) *The intersection of all the closed fuzzy sets in A is called the* **closure** *of A, and denote by $\overline{A}$, $cl_X A$ or $cl A$. Hence,*

$$\overline{A} = \cap\{F \in CF(X) : A \subset F\}.$$

It is clear that A is the largest open fuzzy set contained in A and $(\mathring{\mathring{A}}) = A$; $\overline{A}$ is the smallest closed set containing A and $\overline{(\overline{A})} = \overline{A}$.

Definition 2.6[6]. *Let X and Y be fts's, and let $x_\lambda \in F_p(X)$. Then a mapping $f : X \to Y$ is said to be* **fuzzy continuous** *(simply.* **F - continuous***) at x_λ if for each open neighborhood V of $f(x_\lambda)$, there exists an open neighborhood U of x_λ such that $f(U) \subset V$. And the mapping f is said to be fuzzy continuous (on X) if it is F - continuous at each $x_\lambda \in F_p(X)$.*

Lemma 2.C[6]. *Let X and Y be fts's and let $f : X \to Y$ be a mapping. Then the following statement are equivalent :*

(1) *f is F - continuous.*
(2) *For each $V \in OF(Y), f^{-1}(V) \in OF(X)$.*
(3) *For each $F \in CF(Y), f^{-1}(F) \in CF(X)$.*

Definition 2.7[2]. *Let X and Y be fts's. Then a mapping $f : X \to Y$ is said to be :*

(1) **fuzzy open** *(simply,* **F - open***) if $f(U) \in OF(Y)$ for each $U \in OF(X)$.*
(2) **fuzzy closed** *(simply,* **F - closed** *) if $f(F) \in CF(Y)$ for each $F \in CF(X)$.*

Lemma 2.D[6]. *Let X and Y be fts's and let $f : X \to Y$ be any mapping. Then f is F - open if and only if $f(intA) \subset intf(A)$ for each $A \in I^X$.*

Definition 2.8. *A mapping $f : X \to Y$ is said to be* **fuzzy strongly continuous** *if for each $A \in I^X, f(clA) \subset f(A)$ i.e. $f(clA) = f(A)$.*

Definition 2.9. *Let X be a fuzzy topological space and let $A \in I^X$. Then*

(1) *A is said to be a* **regular open fuzzy set** *in X if $A = int(clA)$.*
(2) *A is said to be a* **regular closed fuzzy set** *in X if $A = cl(intA)$.*

We denote the family of all the regular open (resp. regular clsoed) fuzzy sets in X as $ROF(X)$ (resp. $RCF(X)$).

It is clear the $ROF(X) \subset OF(X) and RCF(X) \subset CF(X)$.

3. Fuzzy Completely Continuous Mappings.

Definition 3.1. *A mapping* $f : X \to Y$ *is said to be* **fuzzy completely continuous** *if for each* $O \in OF(Y), f^{-1}(O) \in ROF(X)$.

Obviously, every fuzzy strongly continuous mapping is fuzzy completely continuous and every fuzzy completely continuous mappings is F - continuous. The converce implicatins do not hold as shown by the following examples.

Example 3.2. *Let* $X = \{a, b, c, d\}$ *and let* $\mathcal{T} = \{\underline{1}_X, \underline{0}_X, O_1, O_2, O_3, O_4, O_5\}$,

$$\begin{aligned} \text{where} \quad O_1 &= \{(a, 0.3), (b, 0.6), (c, 0.8), (d, 0)\}, \\ O_2 &= \{(a, 0), (b, 0), (c, 0.8), (d, 0)\}, \\ O_3 &= \{(a, 0.3), (b, 0.6), (c, 0), (d, 0)\}, \\ O_4 &= \{(a, 0.3), (b, 0), (c, 0.8), (d, 0)\}, \\ O_5 &= \{(a, 0.3), (b, 0), (c, 0), (d, 0)\}. \end{aligned}$$

Let $Y = \{p, q, r\}$ *and let* $\mathcal{U} = \{\underline{1}_Y, \underline{0}_Y, O_6, O_7, O_8\}$, *where*

$$\begin{aligned} O_6 &= \{(p, 0.3), (q, 0), (r, 0)\}, \\ O_7 &= \{(p, 0.3), (q, 0.7), (r, 0)\}, \\ O_8 &= \{(p, 0.3), (q, 0.8), (r, 0)\}. \end{aligned}$$

Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ *be the mapping defined by* $f(a) = p$ *and* $f(b) = f(c) = f(d) = r$. *Then* f *is fuzzy completely continuous but not fuzzy strongly continuous.*

Example 3.3. *Let* $X = \{a, b, c, d\}$ *and let* $\mathcal{T} = \{\underline{1}_X, \underline{0}_X, O_1\}$ *where* $O_1 = \{(a, 0.6), (b, 0.6), (c, 0), (d, 0)\}$. *Let* $Y = \{p, q\}$ *and let* $\mathcal{U} = \{\underline{1}_Y, \underline{0}_Y, O_2\}$ *where* $O_2 = \{(p, 0.6), (q, 0)\}$. *Let* $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ *be the mapping defined by* $f(a) = f(b) = p$ *and* $f(c) = f(d) = q$. *Then* f *is F - continuous but not fuzzy completely continuous.*

The restriction of a fuzzy completely continuous mapping may fail to be fuzzy completely continuous as shown by the following example.

Example 3.4. *Let* $(X, \mathcal{T})$ *be the fuzzy topological space. Let* $X = \{a, b, c, d\}, Y = \{p, q, r\}, \mathcal{T} = \{\underline{1}_X, \underline{0}_X, O_1, O_2, O_3\}$ *and let* $\mathcal{U} = \{\underline{1}_Y, \underline{0}_Y, O_4\}$ *where,*

$$\begin{aligned} O_1 &= \{(a, 0.3), (b, 0.6), (c, 0.8), (d, 0)\}, \\ O_2 &= \{(a, 0), (b, 0), (c, 0.8), (d, 0)\}, \\ O_3 &= \{(a, 0.3), (b, 0.6), (c, 0), (d, 0)\}, \\ O_4 &= \{(p, 0.3), (q, 0.6), (r, 0)\}. \end{aligned}$$

Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be the mapping defined by $f(a) = p$, $f(b) = q$, $f(c) = f(d) = r$. Then f is fuzzy completely continuous. But the restriction of f to the set $\{a, d\}$ is not fuzzy completely continuous.

Theorem 3.5. *If $f : X \to Y$ is fuzzy completely continuous and $g : Y \to Z$ is F - continuous, then $g \circ f : X \to Z$ is fuzzy completely continous.*

Proof. Let $U \in OF(Z)$. Since g is fuzzy F - continuous, $g^{-1}(U) \in OF(Y)$. Since f is fuzzy completely continuous, $f^{-1}(g^{-1}(U)) \in ROF(X)$. But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. So $(g \circ f)^{-1}(U) \in ROF(X)$. Hence $g \circ f$ is fuzzy completely continuous. ■

Corollary 3.6. *The composite of two fuzzy completely continuous mappings is fuzzy completely continuous.*

Definition 3.7. *A mapping $f : X \to Y$ is said to be* **fuzzy almost open** *if for each $A \in ROF(X), f(A) \in OF(Y)$.*

Theorem 3.8. *Let $f : X \to Y$ be fuzzy almost open, fuzzy completely continuous and surjective. If $g : Y \to Z$ is the mapping such that $g \circ f$ is fuzzy completely continuous, then g is F - continuous.*

Proof. Let $G \in OF(Z)$. Since $g \circ f$ is fuzzy completely continuous, $(g \circ f)^{-1}(G) \in ROF(X)$. Since f is fuzzy almost open, $f((g \circ f)^{-1}(G)) \in OF(Y)$. But $(g \circ f)^{-1}(G) = (f^{-1} \circ g^{-1})(G)$. Since f is surjective, $f((g \circ f)^{-1}(G)) = g^{-1}(G)$. So $g^{-1}(G) \in OF(Y)$. Hence g is F - continuous. ■

Corollary 3.9. *Let $f : X \to \Pi_{\alpha \in \Lambda} X_\alpha$ be fuzzy completely continuous. For each $\alpha \in \Lambda, f_\alpha : X \to X_\alpha$ be the mapping defined by $f_\alpha(x) = x_\alpha$ if $f(x) = (x_\alpha)$. Then f_α is fuzzy completely continuous for each $\alpha \in \Lambda$.*

Definition 3.10[1]. *A fts X is said to be* **fuzzy mildly normal** *if for each $F_1, F_2 \in RCF(X)$ such that $F_1 \odot F_2 = \underline{O}_X$,there exist $U, V \in OF(X)$ such that $U \odot V = \underline{O}_X, F_1 \subset U$ and $F_2 \subset V$, where $(U \odot V)(x) = \max[0, A(x) + B(x) - 1]$ for each $x \in X$.*

Lemma 3.A[1,9]. *Let X be a fts. Then the followings are equivalent :*

(1) *X is fuzzy mildly normal.*
(2) *For each $A \in RCF(X)$ and each $B \in RCF(X)$ containing A, there exists $V \in OF(X)$ such that $A \subset V \subset clV \subset B$.*

(3) *For each* $B \in ROF(X)$ *containing* $A \in RCF(X)$, *there exists* $U \in ROF(X)$ *such that* $A \subset U \subset clU \subset B$.
(4) *Foe each* $A, B \in RCF(X)$ *such that* $A \odot B = \underline{0}_X$, *there exist* $U, V \in OF(X)$ *such that* $A \subset U, B \subset V$ *and* $clU \cap clV = \underline{0}_X$.

Definition 3.11[4]. *A fts X is said to* **fuzzy normal** *if for each* $F \in CF(X)$ *and each* $U \in OF(X)$ *such that* $F \subset U$, *there exists a* $V \in I^X$ *such that* $F \subset intV \subset clV \subset U$.

Theorem 3.12. *Let X be a fuzzy mildy normal space and let Y be a fts. If* $f : X \to Y$ *is fuzzy completely continuous, fuzzy closed,open and surjective, then Y is fuzzy normal .*

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