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Uniform ultimate boundedness of control systems with matched and mismatched uncertainties by Lyapunov-like method

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Abstract The recently proposed control method using a Lyapunov-like function can give global asymptotic stability to a system with mismatched uncertainties if the uncertainties are bounded by a known function and the uncontrolled system is locallly and asymptotically stable. In this paper, we modify the method so that it can be applied to a system not satisfying the latter condition without deteriorating qualitative performance. The assured stability in this case is uniform ultimate boundedness which is as useful as global asymptotic stability in the sense that it is global and the bound can be taken arbitrarily small. By the proposed control law we can deal with both matched and mismatched uncertain systems. The above facts conclude that Lyapunov-like control method is superior to any other Lyapunov direct methods in its applicablity to uncertain systems.

Keywords Lyapunov-like function, Lyapunov direct method, Uniform ultimate boundedness, Global asympotic stability, mismatched uncertainties

1. INTRODUCTION

There are several control methods for a system with uncertainties which use Lyapunov direct method. These methods use the stability of the nominal system for stabilizing an uncertain system. In the methods, we can design the control law in the following way: we first select a Lyapunov function of the nominal system and regard the Lyapunov function as a Lyapunov function of the whole system with uncertainties and then we design the control law so that the Lyapunov function is decreased along all the solution trajectories of the uncertain system. The uncertainties are classified into matched and mismatched uncertainties. The min-max method[2] and the saturation type method[1] are representative for a system with matched uncertainties, where a norm is used to estimate the uncertainties. If a known function exists which is always larger than or equal to the uncertainty function when it is measured by the norm, the uncertain system can be stabilized by those control methods. But the control methods may make the closed loop system unstable when it is applied to a system with mismatched uncertainties. This depends on the selected Lyapunov function and it is very difficult to find the Lyapunov function by which a bounded control law can be designed. For a system with equivalently matched uncertainties, it is possible to find such a Lyapunov function[3]. But considerably limited class of uncertain systems have equivalently matched uncertainties.

Recently the nonlinear dynamic control method[4] and the control method using a Lyapunov-like function[5] have been proposed which can be applied to a system with mismatched uncertainties not satisfying the conditions such as the equivalently matched condition. In the nonlinear dynamic control method, a bounded control law is designed to stabilize a system with mismatched uncertainties by generating new state governed by nonlinear equation dependent on system states. And in the Lyapunov-like control method, a bounded control law is designed by using the Lyapunov-like function that works locally as a Lyapunov function. The stability condition of the nominal system is more restricted in the Lyapunov-like control method than in the nonlinear dynamic control method. But when we consider the condition for the uncertainties the restriction is looser in the Lyapunov-like control method than in the nonlinear dynamic method: the uncertainties in the Lyapunov-like control method is sufficient to be norm-bounded by a known function.

In [5] local asymptotic stability of the uncontrolled system is assumed to globally and asymptotically stabilize a system with mismatched uncertainties. Also, control law works only during initial period until the states reach the region of attraction of the uncertain system. Thus the Lyapunov-like control method is not available for an oscillating control. In this paper we modify the control law of [5] and prove that a system with mismatched uncertainties can be uniformly and ultimately bounded without assuming the local asympotic stability of the uncontrolled system. We also show the proposed method can be applied to a system with matched and mismatched uncertainties under the same condition that the uncertainties are norm-bounded.

2. LYAPUNOV-LIKE FUNCTION

We consider the following nonlinear system with mismatched uncertainties.

$$\dot{x}(t) = Ax(t) + Bu(t) + \xi(x(t)) \qquad (1)$$

where $t \in R$ is time, $x(t) \in R^n$ is the state, $u(t) \in R$ is the input and $\xi(x(t)) \in R^n$ is the mismatched uncertainty.

In eq.(1) we call the system with $\xi(\cdot) = 0$ the nominal system and the system with $u(\cdot) = 0$ the uncontrolled system. If the uncontrolled system of eq.(1) is locally and asymptotically stable, the solution trajectories originating at the initial states in the region of attraction converge to the equilibrium point. But the solution diverses when the initial states start at the point out of the region of attraction.

To stabilize globally and asymptotically the system of eq.(1), it is suffient to find such a Lyapunov function $V(x(t)) = x(t)^T P x(t)$ (P is positive definite) as $(\nabla_x V(x(\cdot)))B \neq 0$ is satisfied. But it is very difficult to find such a Lyapunov function. (We hereafter will assume a quadratic function as a Lyapunov function.)

[5] proposes the following Lyapunov-like function.

$$\bar{V}(x(t)) = \begin{cases} V_1(x(t)) & p(x(t)) \leq -\pi/4 \\ V_2(x(t)) & p(x(t)) > -\pi/4 \end{cases} \quad (2)$$

$$V_1(x(t)) = qV(x(t)) \quad (3)$$

$$V_2(x(t)) = qV(x(t)) + cos(p(x(t)) - \pi/4) \quad (4)$$

where

$$p(x(t)) = (\nabla_x V(x(t)))B \quad (5)$$

$$q = (\nabla_x p(x(t)))B \quad (6)$$

(q is positive interger) and

$$c_1 \parallel x(t) \parallel^2 \leq V(x(t)) \leq c_2 \parallel x(t) \parallel^2 \quad (7)$$

The Lyapunove-like function of eq.(2) satisfies with respect to eq.(1)

$$\nabla_x \bar{V}(x(t))B \neq 0$$

The control law using the Lyapunov-like function of eq.(2) in [5] gives global asymptotic stability for an example system of eq.(1). But it requires the uncontrolled system of eq.(1) to be asymptotically stable in the neighborhoood of the equilibrium point. Because of the requirement the applicable systems of the control law are restricted to the systems that the uncontrolled systems are locally and asymptotically stable or stabilizable. Moreover, [5] requires that the region of attraction of the uncontrolled system should be known and that a certain condition related to the region should be satisfied; thus a Lyapunov function should be selected for the condition to be satisfied. However it becomes more difficult to select such a Lyapunov function as the dimension of a target system gets larger.

To solve the above problems, we modify the control law of [5] as eq.(8).

$$u(t) = \begin{cases} u_1(x(t)) & for \quad p(x(t)) \leq -\pi/4 \\ u_2(x(t)) & for \quad p(x(t)) > -\pi/4 \end{cases} \quad (8)$$

$u_1(x(t))$ and $u_2(x(t))$ are

$$u_i(x(t)) = -\frac{\beta_i(x(t)) + \rho_i(x(t))}{\parallel \alpha_i(x(t)) \parallel} \frac{\mu_i(x(t))}{\parallel \mu_i(x(t)) \parallel} \quad (9)$$

$$\mu_i(x(t)) = \frac{\beta_i(x(t)) + \rho_i(x(t))}{\parallel \alpha_i(x(t)) \parallel} \alpha_i(x(t))^T$$

where $\alpha_i(x(t)) = (\nabla_x V_i(x(t)))B$, $\rho_i(x(t)) = \parallel \nabla_x V_i(x(t)) \parallel \cdot \eta(x(t))$, $\beta_i(x(t)) = \nabla_x V_i(x(t))Ax(t) + c_3 \parallel x(t) \parallel^2$ $(i = 1, 2)$

By the control law of eq.(8) the solution of eq.(1) can be uniformly and ultimately bounded within the neigborhood of the equilibrium point. As we can see in theorem1 uniform and ultimate boundedness is similar to global asymptotic stability because it is global and the bound can be taken arbitrarily small. The difference between the control law of eq.(8) and that of [5] is as follows: the control law of [5] works only during the period while the states are brought into the region of attraction and then the convergence of the states is taken over to the stability of the uncontrolled system. On the other hand the control law of eq.(8) works all the time.

3. UNIFORM ULTIMATE BOUNDEDNESS

We rewrite here the same assumptions as in the control for a system with matched uncertainties.

(Assumption 1)
There exists a continuous function $\eta(x(t))$ satisfying

$$\parallel \xi(x(t)) \parallel \leq \eta(x(t))$$

(Assumption 2)
The nominal system of eq.(1) is controllable.

Lemma 1
Let $x(\cdot)$ be the solution of the closed loop system consisted of eq.(8) and eq.(1) for the initial state of $x(t_0) = x_0$. Then $\parallel x(t) \parallel \leq \delta(r)$ is satisfied for $\parallel x_0 \parallel \leq r$ and $t \geq t_0$ where r is a positive integer and

$$\delta(r) = \begin{cases} [\frac{c_2}{c_1}\epsilon^2 + \frac{2}{qc_1}]^{1/2} & if \quad r \leq \epsilon \\ [\frac{c_2}{c_1}r^2 + \frac{2}{qc_1}]^{1/2} & if \quad r > \epsilon \end{cases} \quad (10)$$

$$\epsilon > (\frac{2}{qc_1})^{1/2} \quad (11)$$

(Proof)
First, differentiate $\bar{V}(x(t))$ along the solution trajectories of eq.(1), then

$$\frac{d\bar{V}(x(t))}{dt} = (\nabla_x \bar{V}(x(t)))Ax + (\nabla_x \bar{V}(x(t)))Bu(t)$$
$$+(\nabla_x \bar{V}(x(t)))\xi(x(t))$$

is obtained. Substituting eq.(8) gives

$$\leq \nabla_x V_i(x(t))Ax(t) + \rho_i(x(t))$$
$$+(\nabla_x V_i(x(t)))B \cdot \frac{-(\beta_i(x(t)) + \rho_i(x(t)))}{\parallel (\nabla_x V_i(x(t)))B \parallel} \frac{\mu_i(x(t))}{\parallel \mu_i(x(t)) \parallel}$$
$$= \nabla_x V_i(x(t))Ax(t) + \parallel \rho_i(x(t)) \parallel + \frac{- \parallel \mu_i(x(t)) \parallel^2}{\parallel \mu_i(x(t)) \parallel}$$

and

$$\frac{d\bar{V}(x(t))}{dt} \leq -c_3 \parallel x(t) \parallel \quad (12)$$

Next, consider the solution $x(t)$ of eq.(1) with the initial value $x(t_0) = x_0 (\leq r)$ and put

$$\hat{r} = max\{r, \epsilon\}$$

then $\parallel x_0 \parallel \leq \hat{r}$ and $\epsilon \leq \hat{r}$ are satisfied.

Then from eq.(10) we can rewrite $\delta(r)$ of eq.(10) as

$$\delta(r) = \{\frac{c_2}{c_1}\hat{r}^2 + \frac{2}{qc_1}\}^{1/2}$$

By using $c_1 \parallel x(t) \parallel^2 \leq V(x(t)) \leq c_2 \parallel x(t) \parallel^2$ and thus using

$$(qc_1)\hat{r}^2 \leq (qc_2)\hat{r}^2$$

we obtain

$$\hat{r} \leq (\frac{c_2}{c_1})^{1/2}\hat{r} < \delta(r)$$

Thus

$$\parallel x(t_0) \parallel = \parallel x_0 \parallel \leq \hat{r} < \delta(r)$$

is derived.

Finally, let us assume for $t_3 > t_0$

$$\| x(t_3) \| > \delta(r) \tag{13}$$

then since $x(\cdot)$ is continuous and

$$\| x(t_0) \| \leq \hat{r} < \delta(r) < \| x(t_3) \|$$

there must exists $t_2 \in [t_0, t_3)$ such that $\| x(t_2) \| = \hat{r}$ and

$$\| x(t) \| \geq \hat{r}, \quad t \in [t_2, t_3)$$

are satisfied. But from eq.(2)

$$-1 + qc_1 \| x(t_3) \|^2 \leq V(x(t_3))$$

is derived. Using eq.(12) gives

$$-1 + qc_1 \| x(t_3) \|^2 < qc_2 \| x(t_2) \|^2 + 1$$

and from which

$$\| x(t_3) \| < [\frac{c_2}{c_1}\hat{r}^2 + \frac{2}{qc_1}]^{1/2} = \delta(r)$$

is obtained. This is contradict to the assumption of eq.(13). Therefore

$$\| x(t) \| \leq \delta(r), \quad t \geq t_0$$

is assured.

We will show in the following theorem1 the solution of eq.(1) to be uniformly and ultimately bounded by using the control law of eq.(8).

Theorem 1

For a ϵ satisfying eq.(11) and $r(> \| x_0 \|)$, $T(\epsilon, r) < \infty$ exists and the solution of the closed loop system of eq.(1) and eq.(8) becomes $\| x(t) \| \leq \bar{\delta}$ at $t \geq t_0 + T(\epsilon, r)$ where

$$\bar{\delta} = [\frac{c_2}{c_1}\epsilon^2 + \frac{2}{qc_1}]^{1/2} \tag{14}$$

$$T(\epsilon, r) = \frac{qc_1\psi^2 + 2}{c_3\delta_2^2} \tag{15}$$

$$c_2 r^2 < c_1\psi^2, \quad \psi < \infty \tag{16}$$

$$qc_2\delta_2^2 < min\{qc_1\epsilon^2 - 2, qc_2 r^2\} \tag{17}$$

(proof)

For the initial value of $\| x(t_0) \| < r$ and the initial time t_0, a time $t_1 \in [t_0, t_0 + T(\epsilon, r)]$ exists such that $\| x(t_1) \| < \delta_2$ is satisfied. This is proved by contradiction as follows. First, let asssume

$$\| x(t) \| \geq \delta_2, \quad t \in [t_0, t_0 + T(\epsilon, r)] \tag{18}$$

then from eq.(2) and eq.(18)

$$-1 + qc_1\delta_2^2 \leq \bar{V}(x(t_0 + T(\epsilon, r)))$$

$$= V(x(t_0)) + \int_{t_0}^{t_0+T(\epsilon,r)} \dot{V}(x(\tau))d\tau$$

and by using $\| x(t_0) \| < r$ and eq.(12)

$$\leq 1 + qc_2 r^2 - \int_{t_0}^{t_0+T(\epsilon,r)} c_3 \| x(\tau) \|^2 d\tau$$

and thus using eq.(18)

$$\leq 1 + qc_2 r^2 - c_3\delta_2^2 T(\epsilon, r)$$

and finally using eq.(15)

$$= -1 + qc_2 r^2 - qc_1\psi^2$$

are derived. From which

$$qc_1\delta_2^2 \leq qc_2 r^2 - qc_1\psi^2$$

is derived and by using eq.(16)

$$qc_1\delta_2^2 < 0$$

is derived. But this is a contradiction and thus shows that eq.(18) is true.

Next, let $t' = t_0 + T(\epsilon, r)$ then from $\dot{\bar{V}}(x(t)) \leq 0$

$$-1 + qc_1 \| x(t') \|^2 \leq \bar{V}(x(t')) \leq \bar{V}(x(t_1))$$

is derived and from eq.(2)

$$-1 + qc_1 \| x(t') \|^2 \leq qc_2 \| x(t_1) \|^2 + 1$$

is derived. Using $\| x(t_1) \| < \delta_2$ leads to

$$-1 + qc_1 \| x(t') \|^2 \leq qc_2 \| \delta_2 \|^2 + 1$$

and using eq. (17) results in

$$-1 + qc_1 \| x(t') \|^2 < 1 + qc_1\epsilon^2 - 2$$

Thus

$$c_1 \| x(t') \|^2 < c_1\epsilon^2$$

and

$$\| x(t') \| < \epsilon$$

are derived.

Finally, since $t' = t_0 + T(\epsilon, r)$ and $\| x(t') \| < \epsilon$, from lemma1 $\| x(t') \| \leq \delta(r)$ and thus

$$x(t) \leq \bar{\delta} = [\frac{c_2}{c_1}\epsilon^2 + \frac{2}{qc_1}]^{1/2}, \quad t \geq t_0 + T(\epsilon, r)$$

are derived and the proof is completed.

4. SIMULATION

We examine the result of the previous section by the following numerical example.

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0.1x_1(t)^2 \\ 0 \end{bmatrix} \tag{19}$$

where $\xi(x(t)) = [0.1x_1(t)^2 \quad 0]^T$ is the mismatched uncrtainties.

Let us take the following as a Lyapunov function candidate for the uncontrolled system of eq.(19).

$$V(x(t)) = x(t)^T P x(t) = x(t)^T \begin{bmatrix} 7 & 1 \\ 1 & 2 \end{bmatrix} x(t) \tag{20}$$

Then $p(x(t)) = 2x_1(t) + 4x_2(t)$, $q = 4$. By using this we can make $\bar{V}(x(t))$ as in eq.(2).

For eq.(1) and $\bar{V}(x(t))$, the following input variables are obtained.

$\alpha_1(x(t)) = q * (2x_1(t) + 4x_2(t))$

$\alpha_2(x(t)) = q * (2x_1(t) + 4x_2(t)) - q * sin(p(x(t)) - \pi/4)$

$\rho_1(x(t)) = \| \nabla_x V_1(x(t)) \| *0.1(1 + x_1(t)^2)$

$\rho_2(x(t)) = \rho_1(x(t)) + 0.2 * (1 + x_1(t)^2)$

$$\beta_1(x(t)) = 2q * x(t)^T P \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \| x(t) \|^2$$

$\beta_2(x(t)) =$

$$\beta_1(x(t)) - sin(p(x(t) - \pi/4)\nabla_x p(x(t)) * \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t)$$

Using those, we design the control law of eq.(8). If we set $r = 2.5$, $\epsilon = 0.70$ then since $c_1 = 1.81$, $c_2 = 7.20$ the coefficients of theorem1 become

$\psi = 4.48$, $\delta_2 = 0.23$

$T(\epsilon, r) = 95.13$

$\bar{\delta} = 1.50$.

The simulation results are as follows. First, the system response for the initial value of $x_0 = [2 \;\; 1]^T$ and $u(\cdot) = 0$ is shown divergent as in Fig. 4.1.

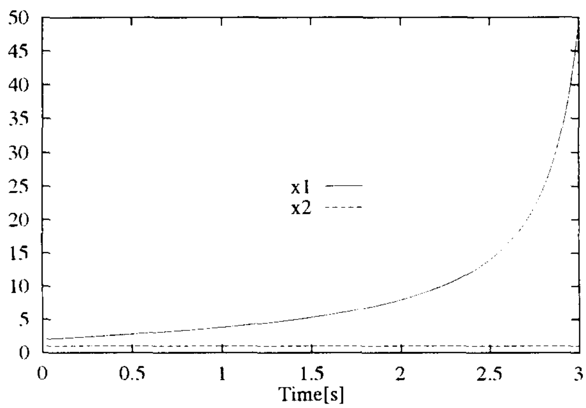

Fig. 4.1 Response for zero input

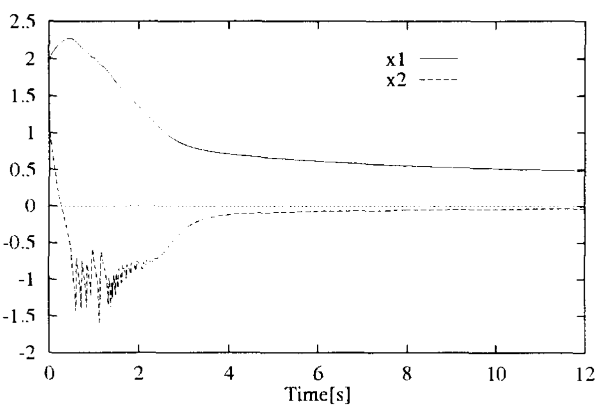

Fig. 4.2(a) Response for Lyapunov-like control input

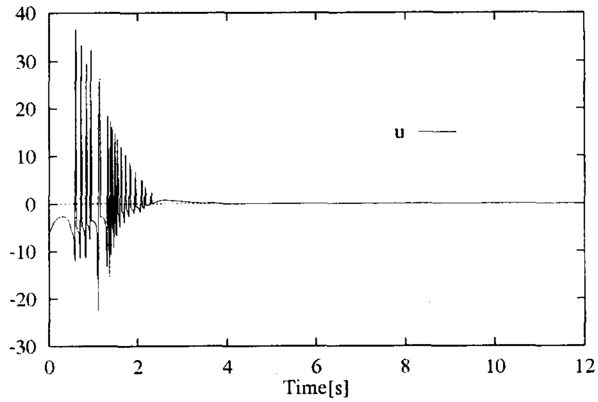

Fig. 4.2(b) Control input

On the other hand, when the Lyapunov-like control law is introduced, $\| x(t) \| \leq 0.70 (t \geq 10)$ as in Fig. 4.2(a), and thus the result is as predicted in the previous section. Also we can see that the estimation of $T(\epsilon, r)$ in theorem1 is conservative.

5. CONCLUSIONS

In this paper we modified the Lyapunov-like control law of [5] and could give to a system with mismatched uncertainties uniform and ultimate boundedness under the same conditions as assumed in a system with matched uncertainties. This achievement can apply Lyapunov-like control law to oscillating control problem.

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