최소 엔트로피를 고려한 혼합 H_2/H_∞ 제어기 구성

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Mixed H_2/H_∞ Controller Design Considering Minimum Entropy

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Abstract: In this paper, we represented the relation of minimum entropy/ H_∞-controller and mixed H_2/H_∞-controller. An H_2 controller design problem involving a constraint on H_∞ disturbance attenuation is considered. By the equivalence of the mixed H_2/H_∞ control problem and the minimum entropy/ H_∞-control problem, we presented the controller state-space realization. Decentralized case was illustrated briefly.

1. Introduction

Entropy has established itself as an important notation, with a wide applicability in a number of diverse subjects, information measure, used in spectral analysis. Mustafa showed the entropy of a system which satisfies an H_∞-norm bound, and derive some important properties, entropy is an upper bound on H_2 cost, and interpret the H_∞-norm bound on the system as proving a proving a prespecified level of robustness.

Recently mixed H_2 and H_∞ optimal control problems have received a great deal of attention(Bernstein and Haddad, 1989; Mustafa, 1989; Mustafa and Glover, 1988; Mustafa and Glover, 1990). Mustafa and Glover(1988) solved the problem of maximizing the entropy of a stabilized closed-loop system. The solution exploits the parameterization of all closed-loop systems that meet an H_∞-norm bound. And they showed that the central solution of this set is shown to maximize the entropy at infinity. Bernstein and Haddad(1989) considered the case of one exogeneous input and two observed outputs. They used Lagrange multiplier technique, and under the assumption that the order of the controller is specified, they derived a necessary condition for minimizing an upper bound of the H_2-norm of one transfer matrix, subject to an H_∞-norm constraint.

In Chapter 2, we formulate the problem, and define the entropy and auxiliary cost. In Proposition 1, we illustrated the controller realization which solves the minimum entropy/ H_∞-control problem. Decentralized mixed problem is extended in Chapter 3. Some conclusion is contained in Chapter 4.

2. Statement of the Problem

Control problem addressed in this paper concerns the finite-dimensional linear time-invariant feedback system depicted in Fig. 1.

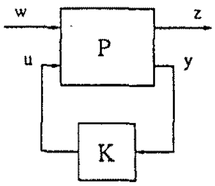

Fig. 1. The closed-loop system.

2.1 centralized case

Consider an n-state plant F with a state space description:

$$\dot{x}(t) = Ax(t) + B_w w(t) + Bu(t) \tag{1}$$

$$z(t) = C_z x(t) + D_{12}u(t) \tag{2}$$

$$y(t) = Cx(t) + D_{21}w(t), \tag{3}$$

where $x(t) \in R^n$ is the state variable, $w(t)$ represent external input. $z(t) \in R^{n_z}$ is the error output, $u(t) \in R^m$ and $y(t) \in R^r$ are the control input and the measured output.

In input-output form, the system can be represented as

$$\begin{bmatrix} z \\ y \end{bmatrix} = P(s) \begin{bmatrix} w \\ u \end{bmatrix},$$

where the transfer function $P(s)$ will be written in packed matrix form:

$$P =: \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} A & B_w & B \\ C_z & 0 & D_{12} \\ C & D_{21} & 0 \end{bmatrix}.$$

In the following, we *assume that*

A1) (A, B) *is stabilizable and* (C, A) *is detectable,*

A2) $D_{12}^T D_{12} = I$, $D_{21} D_{21}^T = I$.

A3) $\begin{bmatrix} A - \lambda I & B \\ C_z & D_{12} \end{bmatrix}$ *and* $\begin{bmatrix} A - \lambda I & B_w \\ C & D_{21} \end{bmatrix}$ *are, respectively,*

full column and row rank for all λ, $\lambda + \bar{\lambda} = 0$.

Connecting an n-state feedback controller $K(s)$, with state-space description:

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \quad (4)$$

$$u(t) = C_c x_c(t). \quad (5)$$

We denote $H(K(s)) = H(s)$ the closed-loop transfer functions of the system when using compensator $K(s)$. Closed-loop transfer function from w to z, as follows:

$$H(s) = \left[\begin{array}{cc} \tilde{A} & \tilde{D} \\ \tilde{E} & 0 \end{array}\right]$$

with state-space description

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{D}w, \quad (6)$$

$$z = \tilde{E}\tilde{x}. \quad (7)$$

$$\tilde{x} = \left[\begin{array}{c} x \\ x_c \end{array}\right], \tilde{A} = \left[\begin{array}{cc} A & BC_c \\ B_c C & A_c \end{array}\right], \tilde{D} = \left[\begin{array}{c} B_w \\ B_c D_{21} \end{array}\right], \tilde{E} = [\, C_z \;\; D_{12} C_c].$$

Definition 1. The *entropy* of the closed-loop transfer function $H(s)$, for a tolerance γ such that $\|H(s)\|_\infty < \gamma$, at an arbitrary point $s_0 \in (0, \infty]$, is defined by

$$I(H,\gamma) = \lim_{s_0 \to \infty} -\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty} \ln|det(I - \gamma^{-2}H^*(j\omega)H(j\omega))| \cdot \left[\frac{s_0}{|s_0 - j\omega|}\right]^2 d\omega$$

where $H^*(s) = H^T(-s)$.

The minimum entropy/H_∞-control problem. Find, for the plant F, a feedback controller K such that:

(i) K stabilizes F.

(ii) The closed-loop transfer function $H(K(s)) = H$ satisfies the H_∞-norm bound $\|H(s)\|_\infty < \gamma$, where $\gamma \in$ R is given.

(iii) The closed-loop entropy $I(H,\gamma)$ is minimized.

Definition 2. The auxiliary cost associated with $H(s)$, where $\|H(s)\|_\infty < \gamma$, is defined by

$$J(H,\gamma) := Tr[Q_s \tilde{E}^T \tilde{E}]$$

where $Q_s > 0$ is the stabilizing solution of the algebraic Riccati equation

$$\tilde{A}Q + Q\tilde{A}^T + \gamma^{-2} Q\tilde{E}^T\tilde{E}Q + \tilde{D}\tilde{D}^T = 0,$$

The mixed H_2/H_∞ control problem. Find, for the plant F, a feedback controller K such that:

(i) K stabilizes F.

(ii) The closed-loop transfer function $H(K(s)) = H$ satisfies the H_∞-norm bound $\|H(s)\|_\infty < \gamma$, where $\gamma \in$ R is given.

(iii) The auxiliary cost $J(H,\gamma)$ is minimized.

Definition 3. Quadratic cost $C(H) = \lim_{t\to\infty} E[z^T(t)z(t)] = \|H\|_2^2$.

Remark 1 Quadratic cost $C(H)$ is achieved $C(H) = Tr[\tilde{Q}\tilde{E}^T\tilde{E}]$ when $\gamma \to \infty$. Where $\tilde{Q}$ is the solution to the Lyapunov equation:

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{D}\tilde{D}^T = 0.$$

The H_2 control problem. Find, for the plant F, a feedback controller K such that:

(i) K stabilizes F.

(ii) The H_2 cost $C(H)$ is minimized.

The entropy gives us a guaranteed upper bound on the actual quadratic cost(Bernstein and Haddad, 1989). H_∞-constrained H_2 control problem is to determine controller (4) and (5) which satisfy the following stability condition, H_∞-constraint, H_2 performance.

Closed loop system stability : closed loop system is asymptotically stable,

H_∞-constraint : closed loop transfer function from w to z satisfies the constraint

$$\|H(s)\|_\infty \leq \gamma \quad (8)$$

where $\gamma > 0$ is given constant.

H_2 performance : functional

$$J(A_c, B_c, C_c) = \lim_{t\to\infty} [x^T R_1 x + u^T R_2 u] \quad (9)$$

is minimized. Where R_1, R_2 are weighting matrix.

2.2 The mixed H_∞/H_2 controller

The solution of minimum entropy/H_∞-control problem, express the controller and minimum value of the entropy in terms of the stabilizing solution to two algebraic Riccati equations. Whereas the solution of the mixed H_2/H_∞-control problem, express via three modified Riccati equations. But by the result of Mustafa (1989), the mixed H_2/H_∞ control problem and the minimum entropy/H_∞-control problem are equivalent, we present the controller state-space realization which solves the above mentioned two problems.

Proposition 1. Controller which solves the minimum entropy/H_∞-control problem takes the following state-space realization:

$$K^\gamma = L_1^\gamma(sI - A + M_1^\gamma C + M_2^\gamma C_z + BL_1^\gamma - M_2^\gamma D_{12} L_1^\gamma)^{-1} M_1^\gamma$$

$$:= \left[\begin{array}{cc} A - M_1^\gamma C - M_2^\gamma C_z - BL_1^\gamma + M_2^\gamma D_{12} L_1^\gamma & M_1^\gamma \\ L_1^\gamma & 0 \end{array}\right].$$

where $M^\gamma = [\, M_1^\gamma \;\; M_2^\gamma] = [\, Y^\gamma C^T + B_w D_{21}^T \;\; -\gamma^{-2} Y^\gamma C_z^T]$,

$$H_Y^\gamma = \left[\begin{array}{cc} A^T - C^T D_{21} B_w^T & \gamma^{-2} C_z^T C_z - C^T C \\ -B_w B_w^T + B_w D_{21}^T D_{21} B_w^T & -A + B_w D_{21}^T C \end{array}\right],$$

$$Y^\gamma = Ric(H_Y^\gamma),$$

$$L^\gamma = \left[\begin{array}{c} L_1^\gamma \\ L_2^\gamma \end{array}\right] = \left[\begin{array}{c} D_{12}^T C_z + B^T X^\gamma \\ -(C + \gamma^{-2} D_{21} B_w^T X^\gamma) \end{array}\right](I - \gamma^{-2} Y^\gamma X^\gamma)^{-1},$$

$$H_X^\gamma = \left[\begin{array}{cc} A - BD_{12}^T C_z & \gamma^{-2} B_w B_w^T - BB^T \\ -C_z^T C_z + C_z^T D_{12} D_{12}^T C_z & -A^T + C_z^T D_{12} B^T \end{array}\right],$$

$$X^\gamma = Ric(H_X^\gamma).$$

Proof: The minimum entropy controller is the *central solution* in the parameterization of all stabilizing controllers which keep $\|H(s)\|_\infty < \gamma$ (Mustafa and Glover, 1990). So comparing the assumption with the references(Mustafa, 1989; Mustafa and Glover, 1990), we can propose the state-space realization as the same form(Seo, *et al.*, 1994).

3. Decentralized Mixed H_2/H_∞ Problem

Now we consider a decentralized case briefly. q-channel stabilizable and detectable plant:

$$\dot{x} = Ax + \sum_{i=1}^{q} B_i u_i + Dw, \tag{10}$$

$$z = Ex, \tag{11}$$

$$y_i = C_i x + D_{2i} w, \quad i \in \{1, 2, \ldots, q\}. \tag{12}$$

where $u_i \in R^{m_i}$ and $y_i \in R^{r_i}$ are the control inputs and the measured outputs of channel i, $i = 1, 2, \cdots, q$, $r = \sum_{i=1}^{q} r_i$, $m = \sum_{i=1}^{q} m_i$.

i-th controller of decentralized controller K_d is

$$\dot{x}_{ci} = A_{ci} x_{ci} + B_{ci} y_i, \tag{13}$$

$$u_i = C_{ci} x_{ci}, \; i = 1, 2, \ldots, q. \tag{14}$$

We denote $H_d(K_d(s)) = H_d$ the closed-loop transfer functions of the system when using compensator $K_d(s)$.

Bernstein and Haddad(1989) considered the solution of the mixed H_2/H_∞ control problem via auxiliary cost $J(H, \gamma)$ minimizing method. So we extend the decentralized case, in this paper, q-channel controller was determined.

Forming the Lagrangian

$$L(A_c, B_c, C_c, Q, P, \lambda) = Tr\{\lambda Q \tilde{E}^T \tilde{E} + [\tilde{A}Q + Q\tilde{A}^T + \beta^2\gamma^{-2} Q \tilde{E}^T \tilde{E} Q + \tilde{D}\tilde{D}^T]\} \tag{15}$$

where the Lagrangian multipliers $\lambda \geq 0$ and $P \in R^{\tilde{n} \times \tilde{n}}$ are not both zero. $\lambda = 1$ can be assumed without loss of generality. Furthermore, P is nonnegative definite.

Setting $\frac{\partial L}{\partial Q} = 0$

$$(\tilde{A} + \beta^2\gamma^{-2} Q \tilde{E}^T \tilde{E})^T P + P(\tilde{A} + \beta^2\gamma^{-2} Q \tilde{E}^T \tilde{E}) + \tilde{E}^T \tilde{E} = 0 \tag{16}$$

Now partition $\tilde{n} \times \tilde{n}$ P and Q into $n \times n$, $n \times n_c$ and $n_c \times n_c$ subblocks as

$$Q = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \; P = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}.$$

where
$$Q_{12} = \begin{bmatrix} Q_{12} \\ Q_{13} \\ \vdots \\ Q_{1q+1} \end{bmatrix}^T, \; Q_2 = \begin{bmatrix} Q_{22} & Q_{23} & \cdots & Q_{2q+1} \\ Q_{32} & Q_{33} & \cdots & Q_{3q+1} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{q+12} & Q_{q+13} & \cdots & Q_{q+1q+1} \end{bmatrix},$$

$$P_{12} = \begin{bmatrix} P_{12} \\ P_{13} \\ \vdots \\ P_{1q+1} \end{bmatrix}^T, \; P_2 = \begin{bmatrix} P_{22} & P_{23} & \cdots & P_{2q+1} \\ P_{32} & P_{33} & \cdots & P_{3q+1} \\ \vdots & \vdots & \ddots & \vdots \\ P_{q+12} & P_{q+13} & \cdots & P_{q+1q+1} \end{bmatrix}.$$

Thus, the stationary conditions are given by

$$\frac{\partial L}{\partial Q} = 0, \; \frac{\partial L}{\partial A_c} = 0, \; \frac{\partial L}{\partial B_c} = 0, \; \frac{\partial L}{\partial C_c} = 0.$$

Expanding $\tilde{A}Q + Q\tilde{A}^T + \beta^2\gamma^{-2} Q \tilde{E}^T \tilde{E} Q + \tilde{D}\tilde{D}^T = 0$ satisfies

$$AQ_1 + Q_1 A^T + B_i C_{ci} Q_{1i+1}^T + Q_{1i+1} C_{ci}^T B_i^T + \beta^2\gamma^{-2} Q_1 E_1^T E_1 Q_1 + DD^T = 0, \tag{17}$$

$$AQ_{1i+1} + Q_{1i+1} A_c^T + B_i C_{ci} Q_2 + Q_1 C_i^T B_{ci}^T + \beta^2\gamma^{-2} Q_1 E_1^T E_1 Q_{1i+1} = 0, \tag{18}$$

$$A_c Q_2 + Q_2 A_c^T + B_{ci} C Q_{1i+1} + Q_{1i+1}^T C_i^T B_{ci}^T + \beta^2\gamma^{-2} Q_{1i+1}^T E_1^T E_1 Q_{1i+1} + B_{ci} D_{2i} D_{2i}^T B_{ci}^T = 0, \tag{19}$$

And expanding equation(16) yields

$$A^T P_1 + P_1 A + C_i^T B_{ci}^T P_{1i+1}^T + \beta^2\gamma^{-2} E_1^T E_1 (P_1 Q_1 + P_1 Q_{1i+1}^T)^T + \beta^2\gamma^{-2} (P_1 Q_1 + P_{1i+1} Q_{1i+1}^T) E_1^T E_1 + E_1^T E_1 = 0 \tag{20}$$

$$A^T P_{1i+1} + P_{1i+1} A_c + C_i^T B_{ci}^T P_2 + P_1 B_i C_{ci} + \beta^2\gamma^{-2} E_1^T E_1 (P_{1i+1}^T Q_1 + P_2 Q_{1i+1}^T)^T = 0 \tag{21}$$

$$A_c^T P_2 + P_2 A_c + P_{1i+1}^T B_i C_{ci} + C_{ci}^T B_i^T P_{1i+1} = 0. \tag{22}$$

$$i = 1, 2, \cdots, q.$$

Conclusions

We illustrated the minimum entropy/H_∞-controller and mixed H_2/H_∞-controller design problem. Both of the controllers are guaranteed the closed-loop stability, H_∞-norm bound and H_2 performance. Using the equivalence of the mixed H_2/H_∞ control problem and the minimum entropy/H_∞-control problem, the controller state-space realization was derived. Decentralized case was briefly illustrated using Lagrangian.

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