On fuzzy pairwise β-continuous mappings

Young-Bin Im* Kuo-Duok Park**

I. Introduction

Kandil[5] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. In [10], Sampath Kumar introduced and studied the concepts of (τ_i, τ_j)-fuzzy semiopen sets, fuzzy pairwise semicontinuous mappings in the fuzzy bitopological spaces. Also, he defined the concepts of (τ_i, τ_j)-fuzzy α-open sets, (τ_i, τ_j)-fuzzy preopen sets, fuzzy pairwise α-continuous mappings and fuzzy pairwise precontinuous mappings in the fuzzy bitopological spaces and studied some of their basic properties.

In this paper, we generalize the concepts of fuzzy β-open sets, fuzzy β-continuous mappings due to Mashhour, Ghanim and Fata Alla[6] into fuzzy bitopological spaces. We first define the concepts of (τ_i, τ_j)-fuzzy β-open sets and then consider the generalizations of fuzzy β-continuous mappings of [6] in the fuzzy bitopological spaces. Characterizations of fuzzy pairwise β-continuous mappings is obtained. Besides many basic results, results related to products and graph of mapping are obtained in the fuzzy bitopological spaces.

II. Preliminaries

For definitions and results not explained in this paper, we refer to the papers[1, 4, 7, 11] assuming them to be well known.

A system (X, τ_1, τ_2) consisting of a set X with two fuzzy topologies τ_1 and τ_2 on X is called a *fuzzy bitoplogical space* (briefly, *fbts*)[5]. Throughout this paper the indices i, j take values in $\{1, 2\}$ and $i \neq j$. $i = j$ gives the known results in fuzzy topological spaces.

Let μ be a fuzzy set of a *fbts* (X, τ_1, τ_2). μ is called a (τ_i, τ_j)-fuzzy semiopen (briefly,

* Dept. of Mathematics, Seonam University

** Dept. of Mathematics, Dongguk University

$(\tau_i, \tau_j)-fso$) set of X if there exists a ν in τ_i such that $\nu \le \mu \le \tau_j$-Cl ν. The complement of a $(\tau_i, \tau_j)-fso$ set is called a (τ_i, τ_j)-fuzzy semiclosed (briefly, $(\tau_i, \tau_j)-fsc$) set[10].

A mapping $f: (X, \tau_1, \tau_2) \to (Y, \eta_1, \eta_2)$ from a *fbts* X to another *fbts* Y is called a fuzzy pairwise semicontinuous (briefly, *fpsc*) mapping, if $f^{-1}(\nu)$ is a $(\tau_i, \tau_j)-fso$ set of X for each η_i-fo set ν of Y. f is called a fuzzy pairwise continuous, briefly, *fpc* mapping, if and only if the induced mappings $f: (X, \tau_k) \to (Y, \eta_k)$ $(k=1,2)$ are fuzzy continuous mappings[10].

Let μ be a fuzzy set of a *fbts* (X, τ_1, τ_2). μ is called a (τ_i, τ_j)-fuzzy α-open (respectively (τ_i, τ_j)-fuzzy α-closed), briefly, $(\tau_i, \tau_j)-f\alpha o$ (respectively $(\tau_i, \tau_j)-f\alpha c$) set of X, if $\mu \le \tau_i$-Int $(\tau_j$-Cl $(\tau_i$-Int $\mu))$ (respectively τ_i-Cl $(\tau_j$-Int $(\tau_i$-Cl $\mu)) \le \mu$). And, μ is called a (τ_i, τ_j)-fuzzy preopen (respectively (τ_i, τ_j)-fuzzy preclosed), briefly, $(\tau_i, \tau_j)-fpo$ (respectively $(\tau_i, \tau_j)-fpc$) set of X, if $\mu \le \tau_i$-Int $(\tau_j$-Cl $\mu)$ (respectively τ_i-Cl $(\tau_j$-Int $\mu) \le \mu$).

Let $f: (X, \tau_1, \tau_2) \to (Y, \eta_1, \eta_2)$ be a mapping from a *fbts* X to another *fbts* Y. f is called a fuzzy pairwise α-continuous (respectively fuzzy pairwise precontinuous), briefly, *fpαc* (respectively *fppc*) mapping, if $f^{-1}(\nu)$ is a $(\tau_i, \tau_j)-f\alpha o$ (respectively $(\tau_i, \tau_j)-fpo$) set in X for each η_i-fo set ν in Y.

III. Fuzzy pairwise β-continuous mappings

Definition 3.1 Let μ a fuzzy set of a *fbts* (X, τ_1, τ_2). μ is called

(i) a (τ_i, τ_j)-fuzzy β-open (briefly, $(\tau_i, \tau_j)-f\beta o$) set of X if $\mu \le \tau_i$-Cl $(\tau_j$-Int $(\tau_i$-Cl $\mu))$,

(ii) a (τ_i, τ_j)-fuzzy β-closed (briefly, $(\tau_i, \tau_j)-f\beta c$) set of X if τ_i-Int $(\tau_j$-Cl $(\tau_i$-Int $\mu))$ $\le \mu$.

From the above definitions it is clear that every $(\tau_i, \tau_j)-fso$ (respectively $(\tau_i, \tau_j)-fsc$)

set or every $(\tau_i, \tau_j)-fpo$ (respectively $(\tau_i, \tau_j)-fpc$) set is a $(\tau_i, \tau_j)-f\beta o$ (respectively $(\tau_i, \tau_j)-f\beta c$) set. The converse of the implications are not true as the following examples show.

Example 3.2 Let μ and ν be fuzzy sets of $X=\{a, b\}$ defined as follows;

$$\mu(a)=0.5,\ \mu(b)=0.3,$$
$$\nu(a)=0.5,\ \nu(b)=0.6.$$

Consider fuzzy topologies $\tau_1=\{0_X, 1_X\}$ and $\tau_2=\{0_X, \mu, 1_X\}$. Then, each fuzzy set of X is a $(\tau_i, \tau_j)-f\beta o$ (respectively $(\tau_i, \tau_j)-f\beta c$) set but not a $(\tau_i, \tau_j)-fso$ (respectively $(\tau_i, \tau_j)-fsc$) set. ■

Example 3.3 Let μ and ν be fuzzy sets of X as in Example 3.2. Consider fuzzy topologies $\tau_1=\{0_X, \nu^c, 1_X\}$ and $\tau_2=\{0_X, \mu, 1_X\}$. Then, ν is a $(\tau_i, \tau_j)-f\beta o$ set but not a $(\tau_i, \tau_j)-fpo$ set, and ν^c is a $(\tau_i, \tau_j)-f\beta c$ set but not a $(\tau_i, \tau_j)-fpc$ set. ■

Theorem 3.4 (i) Any union of $(\tau_i, \tau_j)-f\beta o$ sets is a $(\tau_i, \tau_j)-f\beta o$ set,
(ii) any intersection of $(\tau_i, \tau_j)-f\beta c$ sets is a $(\tau_i, \tau_j)-f\beta c$ set.

It is clear that the intersection (respectively union) of any two $(\tau_i, \tau_j)-f\beta o$ (respectively $(\tau_i, \tau_j)-f\beta c$) sets need not be a $(\tau_i, \tau_j)-f\beta o$ (respectively $(\tau_i, \tau_j)-f\beta c$) set. Even the intersection (respectively union) of a $(\tau_i, \tau_j)-f\beta o$ (respectively $(\tau_i, \tau_j)-f\beta c$) set with a τ_i-fo (respectively τ_i-fc) set may fail to be $(\tau_i, \tau_j)-f\beta o$ (respectively $(\tau_i, \tau_j)-f\beta c$) set.

Example 3.5 Let μ and ν be fuzzy sets of $X=\{a, b\}$ defined as follows;

$$\mu(a)=0.4,\ \mu(b)=0.7,$$
$$\nu(a)=0.7,\ \nu(b)=0.3.$$

Let $\tau_1=\{0_X, \mu, 1_X\}$ and $\tau_2=\{0_X, \nu^c, 1_X\}$ be fuzzy topologies on X. Then μ and ν are $(\tau_i, \tau_j)-f\beta o$ sets but $\mu \wedge \nu$ is not a $(\tau_i, \tau_j)-f\beta o$ set, and $\mu^c \vee \nu^c$ is not a $(\tau_i, \tau_j)-f\beta c$ set in *fbts* (X, τ_1, τ_j). ■

Theorem 3.6 If μ is a $(\tau_i, \tau_j)-f\beta o$ and $(\tau_i, \tau_j)-fsc$ set of a *fbts* X, then μ is a

$(\tau_j, \tau_i)-fso$ set.

Corollary 3.7 If μ is a $(\tau_i, \tau_j)-f\beta c$ and $(\tau_i, \tau_j)-fso$ set of a *fbts* X, then μ is a $(\tau_j, \tau_i)-fsc$ set.

Theorem 3.8 If μ is a $(\tau_i, \tau_j)-f\beta o$ and $(\tau_i, \tau_j)-fac$ set of a *fbts* X, then μ is a τ_i-fc set.

Corollary 3.9 If μ is a $(\tau_i, \tau_j)-f\beta c$ and $(\tau_i, \tau_j)-fao$ set of a *fbts* X, then μ is a τ_i-fo set.

Theorem 3.10 Let (X, τ_1, τ_2) and (Y, τ_1, τ_2) be *fbts*'s such that X is product related to Y[1]. Then the product $\mu \times \nu$ of a $(\tau_i, \tau_j)-f\beta o$ set μ in X and a $(\eta_i, \eta_j)-f\beta o$ set ν in Y, is a $(\sigma_i, \sigma_j)-f\beta o$ set in the fuzzy product space $(X \times Y, \sigma_1, \sigma_2)$, where σ_k $(k=1,2)$ is the fuzzy product topology[11] generated by τ_k and η_k.

Definition 3.11 Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \eta_1, \eta_2)$ be a mapping from a *fbts* X to another *fbts* Y. Then f is called a fuzzy pairwise β-continuous (briefly, *fpβc*) mapping if $f^{-1}(\nu)$ is a $(\tau_i, \tau_j)-f\beta o$ set of a X for each η_i-fo set of Y.

From the above definition it is clear that every *fpsc* mapping or every *fppc* mapping is a *fpβc* mapping. The converse of these implications are not true as the following examples illustrate.

Example 3.12 Let μ and ν be fuzzy sets of X as in Example 3.2. Consider fuzzy topologies $\tau_1 = \{0_X, 1_X\}$, $\tau_2 = \{0_X, \mu, 1_X\}$, $\eta_1 = \{0_X, \mu, \nu, \mu^c, \nu^c, 1_X\}$ and $\eta_2 = \{0_X, \nu, 1_X\}$ and the identity mapping $i_X: (X, \tau_1, \tau_2) \rightarrow (X, \eta_1, \eta_2)$. Then i_X is a *fpβc* mapping but not a *fpsc* mapping. ■

Example 3.13 Let μ and ν be fuzzy sets of $X=\{a, b\}$ defined as follows;

$$\mu(a)=0.7,\ \mu(b)=0.3,$$
$$\nu(a)=0.7,\ \nu(b)=0.7.$$

Consider fuzzy topologies $\tau_1 = \{0_X, \nu^c, 1_X\}$, $\tau_2 = \{0_X, \mu^c, 1_X\}$, $\eta_1 = \{0_X, \mu, 1_X\}$, $\eta_2 = \{0_X, \mu, \nu, 1_X\}$ and the identity mapping $i_X : (X, \tau_1, \tau_2) \rightarrow (X, \eta_1, \eta_2)$. Then i_X is a *fpβc* mapping but not a *fppc* mapping. ■

The following theorem provides several characterization of *fpβc* mappings.

Theorem 3.14 Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \eta_1, \eta_2)$ be a mapping. The following statements are equivalent:

(i) f is a *fpβc* mapping.

(ii) The inverse image of each η_i-*fc* set of Y is a (τ_i, τ_j)-*fβc* set in X.

(iii) $\tau_i\text{-Int}(\tau_j\text{-Cl}(\tau_i\text{-Int}(f^{-1}(\nu)))) \leq f^{-1}(\eta_i\text{-Cl}\,\nu)$ for each fuzzy set ν of X.

(iv) $f(\tau_i\text{-Int}(\tau_j\text{-Cl}(\tau_i\text{-Int}\,\mu))) \leq \eta_i\text{-Cl}(f(\mu))$ for each fuzzy set μ of X.

Theorem 3.15 Let (X_1, τ_1, τ_2), $(X_2, \omega_1, \omega_2)$, (Y_1, η_1, η_2) and $(Y_2, \sigma_1, \sigma_2)$ be *fbts*'s such that X_1 is product related to X_2[1]. Then the product $f_1 \times f_2 : (X_1 \times X_2, \theta_1, \theta_2) \rightarrow (Y_1 \times Y_2, \rho_1, \rho_2)$, where θ_k (respectively ρ_k) is the fuzzy product topology generated by τ_k and ω_k (respectively η_k and ρ_k) $(k=1,2)$, of *fpβc* mappings $f_1 : (X_1, \tau_1, \tau_2) \rightarrow (Y_1, \eta_1, \eta_2)$ and $f_2 : (X_2, \omega_1, \omega_2) \rightarrow (Y_2, \sigma_1, \sigma_2)$ is a *fpβc* mapping.

Theorem 3.16 Let (X, τ_1, τ_2), (X_1, η_1, η_2) and $(X_2, \omega_1, \omega_2)$ be *fbts*'s and $\pi_k : (X_1 \times X_2, \theta_1, \theta_2) \rightarrow X_k$ $(k=1,2)$ be the projection mappings. If $f: X \rightarrow X_1 \times X_2$ is a *fpβc* mapping, then so is $\pi_k \circ f$.

Theorem 3.17 Let $f: (X_1, \tau_1, \tau_2) \rightarrow (X_2, \eta_1, \eta_2)$ be a mapping from a *fbts* X_1 to another *fbts* X_2. Then, if the graph $g : (X_1, \tau_1, \tau_2) \rightarrow (X_1 \times X_2, \theta_1, \theta_2)$ of f defined by $g(x) = (x, f(x))$ is a *fpβc* mapping then f is a *fpβc* mapping.

That converse of above theorem is false, is established by following example.

Example 3.18 Let μ and ν be fuzzy sets of $X = \{a, b\}$ defined as in Example 3.5. Consider fuzzy topologies $\tau_1 = \{0_X, \mu, 1_X\}$, $\tau_2 = \{0_X, \nu^c, 1_X\}$, $\eta_1 = \{0_X, \nu, 1_X\}$ and

$\eta_2 = \{0_X, \mu^c, 1_X\}$ and the identity mapping $i_X : (X, \tau_1, \tau_2) \rightarrow (X, \eta_1, \eta_2)$. Then f is a *fpβc* mapping, $\mu \times \nu$ is a (θ_i, θ_j)-*fo* set of the fuzzy product space $(X \times X, \theta_1, \theta_2)$, where θ_k $(k=1,2)$ is the fuzzy product topology generated by τ_k and η_k. But its graph g is not *fpβc*, since $g^{-1}(\mu \times \nu) = \mu \wedge f^{-1}(\nu) = \mu \wedge \nu$ is not a (τ_i, τ_j)-*fβo* set of X. ■

References

1. K. K. Azad, *On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity*, J. Math. Anal. Appl. **82** (1981), 14-32.
2. A. S. Bin Shahna, *On fuzzy strong semi-continuity and fuzzy pre-continuity*, Fuzzy Sets and Systems **44** (1991), 303-308.
3. ______________, *Mappings in fuzzy topological spaces*, Fuzzy Sets and Systems 61 (1994), 209-213.
4. C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182-190.
5. A. Kandil, *Biproximities and fuzzy bitopological spaces*, Simon Stevin **63** (1989), 45-66.
6. A. S. Mashhour, M. H. Ghanim and M. A. Fath Alla, *On fuzzy non-continuous mappings*, Bull. Cal. Math. Soc. **78** (1986), 57-69.
7. P. M. Pu and Y. M. Liu, *Fuzzy topology I. Neighbourhood structure of a fuzzy point and Moor-Smith convergence*, J. Math. Anal. Appl. **76** (1980), 571-599.
8. ______________, *Fuzzy topology II. Product and quotient spaces*, J. Math. Anal. Appl. **77** (1980), 20-37.
9. S. Sampath Kumar, *On fuzzy pairwise α-continuity and fuzzy pairwise pre-continuity*, Fuzzy sets and Systems **62** (1994), 231-238.
10. ______________, *Semi-open sets, semi-continuity and semi-open mappings in fuzzy bitoplogical spaces*, Fuzzy Sets and Systems **64** (1994), 421-426.
11. C. K. Wong, *Fuzzy topology: Product and quotient theorems*, J. Math. Anal. Appl. **45** (1974), 512-521.