Usual Fuzzy Metric Space and Fuzzy Heine-Borel Theorem

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1. Introduction.

We shall define the usual fuzzy distance between two fuzzy points in $\mathbb{R}$, the set of all real numbers, using the usual distance between two points in $\mathbb{R}$.

Applying the notion of this usual fuzzy distance, we construct the usual fuzzy topology for $\mathbb{R}$, introduce the notions of lower, stationary and upper cover and obtain the fuzzy Heine-Borel theorem.

2. Preliminaries.

Throughout this paper, we always denote the closed unit interval $[0,1]$ by I, while $I_0 =]0,1]$ and $\mathbb{R}^+ = [0,\infty[$.

As is well known, a **fuzzy set** A in a universe X is characterized by a **membership function** μ_A from X to I. The set $\tilde{P}(X)$, called the **fuzzy power set** of X, is the set of all fuzzy sets in X. The symbol $\emptyset$ will be used to denote the empty (fuzzy) set.

Two fuzzy sets A and B are said to be **equal** (denoted $A = B$) if and only if

$$\mu_A(x) = \mu_B(x) \quad \text{for all} \quad x \in X.$$

The **support** of $A \in \tilde{P}(X)$, denoted by $S(A)$, is the ordinary subset of X:

$$S(A) = \{x \in X | \mu_A(x) > 0\}.$$

The union and intersection of $\{A_j \in \tilde{P}(X) | j \in J\}$, denoted by $\cup_{j\in J}(A_j)$ and $\cap_{j\in J}(A_j)$ respectively, are defined by the membership functions

$$\mu_{\cup_{j\in J}A_j}(x) = \bigvee_{j\in J} \mu_{A_j}(x),$$

$$\mu_{\cap_{j\in J}A_j}(x) = \bigwedge_{j\in J} \mu_{A_j}(x) \quad \text{for all} \quad x \in X.$$

The complement of A, denoted by A^c, is defined by

$$\mu_{A^c}(x) = 1 - \mu_A(x) \quad \text{for all} \quad x \in X.$$

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A is said to be **included in** B, denoted by $A \subseteq B$, if and only if

$$\mu_A(x) \leq \mu_B(x) \quad \text{for all} \quad x \in X.$$

When at least one inequality is strict, the inclusion is said to be **strict** and is denoted $A \subset B$. Obviously, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

A **fuzzy point** or a **fuzzy singleton** in a set X is a fuzzy set in X which is zero everywhere except at one point, say x, where it takes a value, say α, in I_0. This fuzzy point is denoted by x_α, and α and x are called its **value** and **support** respectively, that is, x_α is called a **fuzzy point with support** x **and the value** α. The collection of all fuzzy points in X will be denoted by $F_p(X)$.

A fuzzy point x_α is said to **belong to** a fuzzy set A or x_α is an **element** of A, denoted by $x_\alpha \in A$ if and only if $\alpha < \mu_A(x)$. Thus, we obtain easily the following.

Proposition 2.1. *For* $A \in \tilde{P}(X)$ *the followings hold.*

(i) $A = \cup_{x \in S(A)} x_{\mu_A(x)}$.
(ii) $x_{\mu_A(x)} = \cup_{0<\lambda\leq\mu_A(x)} x_\lambda$.
(iii) $\cup_{0<\lambda\leq\mu_A(x)} x_\lambda = \cup_{0<\lambda<\mu_A(x)} x_\lambda$.
(iv) $A = \cup_{x\in S(A)}(\cup_{0<\lambda\leq\mu_A(x)} x_\lambda) = \cup_{x\in S(A)}(\cup_{0<\lambda<\mu_A(x)} x_\lambda)$.

3. Usual Fuzzy Metric Space.

For a metric space (X, D), a **fuzzy distance** $\tilde{D}$ between fuzzy sets A and B in X is defined using D as

$$\mu_{\tilde{D}(A,B)}(\delta) = \bigvee_{\delta = D(u,v)} (\mu_A(u) \wedge \mu_B(v)) \quad \text{for all} \quad \delta \in \mathbb{R}^+$$

([4]). $\tilde{D}$ is a mapping from $[\tilde{P}(X)]^2$ to $\tilde{P}(\mathbb{R}^+)$.

We define a **distance function** $\tilde{d} : F_p(X) \times F_p(X) \to \tilde{P}(\mathbb{R}^+)$ by the restriction of $\tilde{D}$ to $[F_p(X)]^2$. Since each pair (x_α, y_β) in $[F_p(X)]^2$ corresponds to the fuzzy point $D(x,y)_{\alpha\wedge\beta}$ with support $D(x,y)$ and the value $\alpha \wedge \beta$, we can define the usual fuzzy metric as following:

Definition 3.1. *The* **usual fuzzy metric** $\tilde{d} : F_p(\mathbb{R}) \times F_p(\mathbb{R}) \to \tilde{P}(\mathbb{R}^+)$ *is defined by*

$$\tilde{d}(x_\alpha, y_\beta) = |x - y|_{\alpha\wedge\beta} \quad \textit{for every} \quad (x_\alpha, y_\beta) \in F_p(\mathbb{R}) \times F_p(\mathbb{R}).$$

We call the pair $(\mathbb{R}, \tilde{d})$ *the* **usual fuzzy metric space**, *or* **usual fms** *for short.*

Definition 3.2. *For the usual fms* $(\mathbb{R}, \tilde{d})$ *the* **open fuzzy ball** $B(x_\alpha; r_\alpha)$ **with center** x_α **and radius** r_α *is the fuzzy set*

$$B(x_\alpha; r_\alpha) = \cup\{y_\beta \in F_p(\mathbb{R}) : \tilde{d}(x_\alpha, y_\beta) << r_\alpha\},$$

where $|x-y|_{\alpha\wedge\beta} << r_\alpha$ *means that* $|x-y| < r$ *and* $\alpha \wedge \beta < \alpha$. *By 2.1 we see that* $S[B(x_\alpha; r_\alpha)] =]x-r, x+r[$ *and* $\mu_{B(x_\alpha; r_\alpha)}(y) = \alpha$ *for all* $y \in]x-r, x+r[$. *As in the notation of a fuzzy point we denote* $B(x_\alpha; r_\alpha)$ *by* $]x-r, x+r[_\alpha$ *and call it the* **open fuzzy interval with the value** α. *In the followings, we define the general cases.*

Definition 3.3. *A fuzzy set A in* $\mathbb{R}$ *is called an* **open fuzzy interval with the value** α *for* $\alpha \in I_0$ *or simply* **open fuzzy interval** *if and only if* $S(A)$ *is an open interval* $]a, b[$ *and*

$$\mu_A(x) = \begin{cases} \alpha & \text{if } x \in S(A), \\ 0 & \text{otherwise.} \end{cases}$$

In this case, we shall denote it by $]a, b[_\alpha$.

Similarly, we can define the other fuzzy intervals with the value α,

$$[a, b]_\alpha, \quad [a, b[_\alpha, \quad]a, b]_\alpha$$

where $a = -\infty$ *and* $b = \infty$ *are admissible. If* $a = -\infty$, *for instance ,then* $[a, b]_\alpha$ *means* $]-\infty, b]_\alpha$. *Precisely, we call* $[a, b]_\alpha$, $[a, b[_\alpha$, *and* $]a, b]_\alpha$, *the* **closed fuzzy interval, closed-open fuzzy interval** *and* **open-closed fuzzy interval with the value** α, *respectively.*

Definition 3.4. *A fuzzy set A in* $\mathbb{R}$ *is called an* **open fuzzy set** *or simply* **open** *if and only if for every* $x \in S(A)$ *and for every* $0 \le \lambda < \mu_A(x)$ *there exists an* $\epsilon > 0$ *such that*

$$]x-\epsilon, x+\epsilon[_\lambda \subset A.$$

Proposition 3.5. *A fuzzy set A in* $\mathbb{R}$ *is open if and only if it is the union of open fuzzy intervals.*

By 3.5, we have alternative definition of 3.4.

Definition3.4′. *The union of an arbitrary family of open fuzzy intervals is called an* **open fuzzy set** *or simply* **open**.

Definition 3.6. *A fuzzy set is called a* **closed fuzzy set** *or simply* **closed** *if its complement is open.*

By definition, each open fuzzy interval is open. The empty (fuzzy) set $\emptyset$ is open since it is the empty union of open fuzzy intervals.

It is straightforward to show that the union of an arbitrary family of open fuzzy sets is open. In order to prove that the intersection of two open fuzzy sets A, B is open, we divide our proof into three steps.

1st Step. Let A and B be open fuzzy intervals $]a, b[_\alpha$, $]c, d[_\beta$, respectively. Then, $A \cap B$ is either $\emptyset$ or $A \cap B =]e, f[_{\alpha\wedge\beta}$, where $]e, f[=]a, b[\cap]c, d[$. Thus, in this case, $A \cap B$ is open.

2nd Step. Let A be an open fuzzy interval $]a, b[_\alpha$ and let B be a general open fuzzy set. Then there exists the index set J such that $B = \cup_{j\in J} B_j$, where $B_j =]c_j, d_j[_{\alpha_j}$.Thus, we obtain that

$$\begin{aligned} A \cap B &= A \cap (\cup_{j\in J} B_j) \\ &= \cup_{j\in J}(A \cap B_j) \\ &= \cup_{j\in J}(\ ]a, b[_\alpha \cap]c_j, d_j[_{\alpha_j}). \end{aligned}$$

From 1st step, $A \cap B$ is the union of open fuzzy intervals, so that $A \cap B$ is open.

3rd Step. Let A and B be general open fuzzy sets. Then, we obtain that

$$A \cap B = \cup_{j\in J}(A\cap\ ]c_j, d_j[_{\alpha_j}).$$

From 2nd step, we see that for each $j \in J$ $A\cap\ ]c_j, d_j[_{\alpha_j}$ is open. Thus $A \cap B$ is the union of open fuzzy sets, so that it is open.

Therefore, we have proved the following theorem.

Theorem 3.7. *Let τ be the family of all open fuzzy sets in $\mathbb{R}$. Then τ satisfies the followings:*

(i) For each $\alpha \in I$, $]\infty, \infty[_\alpha \in \tau$, where $]-\infty, \infty[_0$ means the empty set $\emptyset$.
(ii) If $\{A_j | j \in J\} \subseteq \tau$, then $\cup_{j\in J} A_j \in \tau$.
(iii) If $A, B \in \tau$, then $A \cap B \in \tau$.

Definition 3.8. *The family τ in the Theorem 3.7 is called the* **usual fuzzy topology** *for $\mathbb{R}$ and the pair $(\mathbb{R}, \tau)$ the* **usual fuzzy topological space**, *or* **usual fts** *for short.*

4. Heine-Borel Theorem.

Definition 4.1. *A fuzzy set A in a linearly ordered set X is said to be* **bounded above, bounded below,** *and* **bounded** *if and only if $S(A)$ is bounded above, bounded below, and bounded, respectively.*

Definition 4.2. *Let (X,τ) be a fts. A family $\mathcal{G}$ of fuzzy sets in X is a* **cover** *of a fuzzy set B in X if and only if*

$$B \subseteq \cup\{A | A \in \mathcal{G}\}.$$

It is an **open cover** *if and only if each member of $\mathcal{G}$ is an open fuzzy set. A* **subcover** *of $\mathcal{G}$ is a subfamily which is also a cover.*

Definition 4.3. *Let $\mathcal{G}$ be a cover of $B \in \tilde{P}(\mathbb{R})$ in a usual fts $(\mathbb{R},\tau)$. An $x \in S(B)$ is called a* **lower covered point** *for $\mathcal{G}$ if it satisfies one of the followings:*

(i) $\mu_G(x) < \mu_B(x)$ for all $G \in \mathcal{G}$,

(ii) if $\mu_G \leq \mu_B(x)$ for all $G \in \mathcal{G}$ and
if there exists $G \in \mathcal{G}$ such that $\mu_G(x) = \mu_B(x)$, then there exists an $\epsilon > 0$ such that

$$\mu_G(y) < \mu_G(x) \text{ for all } y \in \,]x-\epsilon, x+\epsilon[-\{x\}.$$

$\mathcal{G}$ is a **lower cover** *if and only if each member of $S(B)$ is a lower covered point for $\mathcal{G}$. $\mathcal{G}$ is called a* **lower open cover** *if $\mathcal{G}$ is an open cover.*

An $x \in S(B)$ is called a **stationary covered point** *for $\mathcal{G}$ if and only if*

$$\mu_G(x) \leq \mu_B(x) \text{ for all } G \in \mathcal{G}$$

and

$$\mu_G(x) = \mu_B(x) \text{ for some } G \in \mathcal{G}$$

which is not the type of (ii). $\mathcal{G}$ is a **stationary cover** *of B if and only if each member of $S(B)$ is a stationary covered point for $\mathcal{G}$. $\mathcal{G}$ is called a* **stationary open cover** *of B if $\mathcal{G}$ is an open cover.*

An $x \in S(B)$ is called an **upper covered point** *for $\mathcal{G}$ if and only if there exists $G \in \mathcal{G}$ such that $\mu_B(x) < \mu_G(x)$ if $\mu_B(x) < 1$. In case $\mu_B(x) = 1$, of cause, $\mu_B(x) = \mu_G(x)$ for some $G \in \mathcal{G}$ which is not the type of (ii). $\mathcal{G}$ is an* **upper cover** *of B if and only if each member of $S(B)$ is an upper covered point for $\mathcal{G}$. $\mathcal{G}$ is called an* **upper open cover** *of B if $\mathcal{G}$ is an open cover.*

Definition 4.4. *The fuzzy set K in a fts (X,τ) is called a* **fuzzy compact** *if and only if each upper open cover of K has a finite subcover.*

Lemma 4.5. *In a usual fts $(\mathbb{R},\tau)$, the followings hold.*

(i) If U is open, then $S(U)$ is open.

(ii) If F is closed, then $S(F)$ is closed.

Lemma 4.6. *Let $\mathcal{G}$ be an upper open cover of $K \in \tilde{P}(\mathbb{R})$ in the usual fts $(\mathbb{R}, \tau)$ and let $x \in S(K)$. Then there exist an $\epsilon > 0$ and $G \in \mathcal{G}$ such that*

$$]x-\epsilon, x+\epsilon[_{\mu_K(x)} \subset G \text{ if } \mu_K(x) < 1,$$

$$]x-\epsilon, x+\epsilon[_{\mu_K(x)} \subseteq G \text{ if } \mu_K(x) = 1.$$

Heine-Borel Theorem. *If a fuzzy set K in $\mathbb{R}$ is bounded and colsed in the usual fts $(\mathbb{R}, \tau)$, then it is fuzzy compact.*

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