A NOTE ON GENERALIZED COMMON FIXED POINT THEOREMS FOR FUZZY MAPPINGS

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Abstract : In this note, we give generalized common fixed point theorems for sequences of fuzzy mappings on Menger probabilistic metric spaces.

1. Introduction and Preliminaries

Recently, many authors [1-5] presented some fixed point theorems for multi-valued mappings or fuzzy mappings on metric spaces or probabilistic metric spaces.

In this note, we give generalized common fixed point theorems for sequences of fuzzy mappings on Menger probabilistic metric spaces.

A fuzzy set A in a metric space (E, d) is a function from itself into $[0, 1]$ and the function value $A(x)$ denotes the grade of membership of x in A. The α-level set of A, denoted by $(A)_\alpha$, is defined by $(A)_\alpha = \{x | A(x) \geq \alpha\}$ if $\alpha \in (0, 1]$. We denote $W(E)$ the collection of all fuzzy sets A in E such that each $(A)_\alpha$ is nonempty closed set in E. For $A, B \in W(E)$, we denote by $A \subset B$, iff $A(x) \leq B(x)$ for each $x \in E$. If T is a mapping from E into $W(Y)$, where Y is a metric space, then T is called a fuzzy mapping. For each $x \in E$ we let $\{x\}$ be a fuzzy set with a membership function equals to a characteristic function of the set $\{x\}$.

For the sake of convenience, we recall some definitions, terms and notations in probabilistic metric spaces [1,2,3,5,6].

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}$ the set of all positive integers.

Definition 1.1. A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function, if it is nondecreasing and left-continuous with $\inf F(t) = 0$ and $\sup F(t) = 1$.

Definition 1.2. A probabilistic metric space (in short, a PM-space) is an ordered pair $(E, \Im)$, where E is a nonempty set and $\Im$ is a mapping from $E \times E$ into D^+, the set of all distribution functions. We denote the distribution function $\Im(x, y)$ by $F_{x,y}$ and $F_{x,y}(t)$ represents the value of $F_{x,y}$ at $t \in \mathbb{R}$ for each $x, y \in E$. The function $F_{x,y}$ is assumed to satisfy the following conditions :

(PM-1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$;

(PM-2) $F_{x,y}(0) = 0$;

(PM-3) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in \mathbb{R}$;

(PM-4) if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$, then $F_{x,z}(t_1 + t_2) = 1$.

Definition 1.3. A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a t-norm, if it satisfies the following conditions : for any $a, b, c, d \in [0,1]$,

(T-1) $\Delta(a, 1) = a$;

(T-2) $\Delta(a, b) = \Delta(b, a)$;

(T-3) $\Delta(c, d) \geq \Delta(a, b)$ for $c \geq a$ and $d \geq b$;

(T-4) $\Delta(\Delta(a, b,), c) = \Delta(a, \Delta(b, c))$.

Definition 1.4. A Menger PM-space is a triplet $(E, \Im, \Delta)$, where $(E, \Im)$ is a PM-space and Δ is a t-norm satisfying the following triangle inequality

$$F_{x,z}(t_1 + t_2) \geq \Delta(F_{x,y}(t_1), F_{y,z}(t_2)) \text{ for all } x, y, z \in E \text{ and } t_1, t_2 \geq 0.$$

Schweizer and Sklar [6] have proved that if $(E, \Im, \Delta)$ is a Menger PM-space with a continuous t-norm Δ, then $(E, \Im, \Delta)$ is a Hausdorff topological space in the topology τ induced by the family of neighborhoods :

$$\{U_p(\varepsilon, \lambda) : p \in E, \quad \varepsilon > 0, \quad \lambda > 0\},$$

where

$$U_p(\varepsilon, \lambda) = \{x \in E : F_{x,p}(\varepsilon) > 1 - \lambda\}.$$

Definition 1.5. Let $(E, \Im, \Delta)$ be a Menger PM-space with a continuous t-norm Δ. Let $(x_n)_{n=1}^{\infty}$ be any sequence in E. $(x_n)_{n=1}^{\infty}$ is said to be τ-convergent to $x \in E$ (we write $x_n \xrightarrow{\tau} x$), if for any given $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.

$(x_n)_{n=1}^{\infty} \subset E$ is called a τ-Cauchy sequence, if for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$, whenever $n, m \geq N$.

A Menger PM-space $(E, \Im, \Delta)$ is said to be τ-complete, if each τ-Cauchy sequence in E is τ-convergent to some point in E.

Definition 1.6 [5]. Let $(E, \Im, \Delta)$ be a Menger PM-space, $A, B \in W(E)$ then the fuzzy probabilistic distance from A to B is defined as follows ;

$$F_{A,B}(t) = \inf_{\alpha \in (0,1]} F_{(A)_\alpha,(B)_\alpha}(t), \quad t \geq 0,$$

where $F_{C,D}(t) = \sup_{s<t} \Delta(\inf_{x \in C} \sup_{y \in D} F_{x,y}(s), \inf_{y \in D} \sup_{x \in C} F_{x,y}(s))$ for nonempty closed sets C, D and $t \in \mathbb{R}$.

Remark. For $x, y \in E$, $F_{\{x\},\{y\}}(t) = F_{x,y}(t), \quad t \geq 0.$

2. Main results

Now we give our main results, generalized common fixed point theorems for sequences of fuzzy mappings on Menger PM-spaces.

Theorem 2.1. Let $(E, \Im, \Delta)$ be a τ-complete Menger PM-space with a left-continuous t- norm Δ, and $(T_i)_{i=1}^{\infty} : E \to W(E)$ a sequence of fuzzy mappings. Suppose that there exists a constant $k \in (0, 1)$ such that for any $i, j \in \mathbb{N}$ and any $x, y \in E$, $\{u_x\} \subset T_i x$, there exists a $\{v_y\} \subset T_j y$ satisfying

$$F_{\{u_x\},\{v_y\}}(kt) \geq \min\{F_{x,y}(t), \;\; F_{x,u_x}(t), \;\; F_{y,v_y}(t)\}, \;\; t \geq 0.$$

Then there exists an $x_* \in E$ such that $\{x_*\} \subset T_i x_*$ for $i \in \mathbb{N}$.

Theorem 2.2. Let $(E, \Im, \Delta)$ be a τ-complete Menger PM-space with a left-continuous t-norm Δ satisfying $\Delta(t,t) \geq t$, $t \in [0,1]$, and $(T_i)_{i=1}^{\infty} : E \to W(E)$ a sequence of fuzzy mappings. Suppose that there exists a constant $k \in (0,1)$ such that for any $i, j \in \mathbb{N}$ and any $x, y \in E$, $\{u_x\} \subset T_i x$, there exists a $\{v_y\} \subset T_j y$ satisfying

$$F_{\{u_x\},\{v_y\}}(kt) \geq \min\{F_{x,y}(t), F_{x,u_x}(t), F_{y,v_y}(t), F_{x,v_y}(2t), F_{y,u_x}(2t)\}, t \geq 0,$$

Then there exists an $x_* \in E$ such that $\{x_*\} \subset T_i x_*$ for $i \in \mathbb{N}$.

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