FUZZY LIE IDEALS AND FUZZY LIE SUBALGEBRAS

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ABSTRACT. In this paper we tried to apply the concepts of fuzzy set to Lie algebra, to define some concepts related to fuzzy set to discuss their properties and relations among them.

Introduction

Three decades ago, Zadeh[Z] introduced the new concepts of a set which is now a day called the "fuzzy set", to overcome the difficulty in the Cantor's Set Theory. Since Zadeh formulated the notion of a fuzzy set, fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics, and also there have been wide-ranging applications of the theory of fuzzy sets, from the design of robots and computer simulation to engineering and water resources planning. Rosenfeld[R] introduced the fuzzy sets in the realm of group theory. Since then many mathematicians have been involved in extending the concepts and results of abstract algebra to the broader framework of the fuzzy setting. Moreover, Liu in his pioneering paper ([L1]) introduced and studied the concepts of fuzzy subrings and fuzzy ideals in rings. Subsequently these concepts are studied in the references [DKA1,2],[K1,2,3],[L2]. Katsaras and Liu [KL] introduced fuzzy vector spaces and Nanda [N] introduced fuzzy fields and fuzzy linear spaces, etc. Furthermore some new concepts in fuzzy settings were also introduced in those papers. Recently a procedure to construct the fuzzy subrings and fuzzy ideals generated by a finite sum set of a ring are introduced in paper of [DKA1]. However, not all the results of abstract algebra can be fuzzified. For more details, see references [DKA1,2,3].

The purpose of this paper is to extend the concepts and some results of Lie algebra to the framework of fuzzy setting, using the concepts of level subsets.

In this paper, notation 'set' is ordinary set in the sense of Cantor. Let S be a set. A *fuzzy set* μ of S is a function $\mu : S \longrightarrow [0,1]$. Let μ be a fuzzy set of S and $t \in [0,1]$. The set $\{x \in S : \mu(x) \geq t\}$ is called a *level subset* of μ and is symbolized by μ_t. Clearly, $\mu_t \subseteq \mu_s$, whenever $t \geq s$. We call that *inclusion* of fuzzy sets of S is defined as follows ; $\mu \subseteq \sigma \Leftrightarrow \mu(x) \leq \sigma(x)$, for all $x \in S$. Clearly the set of all fuzzy sets in S is complete lattice $\mathcal{L}$ under this ordering. We shall denote the sup and inf in $\mathcal{L}$ by $\cup$ and $\cap$. The least and greatest elements of $\mathcal{L}$ are constant function 0 and 1.

Let S and S' be any two sets and $f : S \longrightarrow S'$ a function. For any fuzzy set μ of S, we define the fuzzy set σ of S' by

$$\sigma(y) = \begin{cases} \sup\{\mu(x) : x \in f^{-1}(y)\} & \text{if } y \in f(S) \\ 0 & \text{if } y \notin f(S), \end{cases}$$

$\sigma(y)$ is called the *image* of μ under f and denoted by $f(\mu)$. For any fuzzy set ν of S', we define the fuzzy set μ of S by $\mu(x) = \nu(f(x))$, for all $x \in S$, $\mu(x)$ is called the *preimage* of ν under f and denoted by $f^{-1}(\nu)$.

Let S and S' be any two sets and let $f : S \longrightarrow S'$ be any function and μ be a fuzzy set of S. A fuzzy set μ of S is said to be f*−invariant* if $f(x) = f(y) \Rightarrow \mu(x) = \mu(y)$, for all $x, y \in S$.

Let $\{\mu_i : i \in I\}$ be fuzzy sets of S. *Union* and *intersection* of fuzzy sets $\{\mu_i : i \in I\}$ are defined by

$$(\bigcup_{i\in I} \mu_i)(x) = \sup(\mu_i(x)) \text{ and } (\bigcap_{i\in I} \mu_i)(x) = \inf(\mu_i(x)).$$

Let $*$ be a binary operation of a set S and let μ, σ be any two fuzzy sets of S. The *product* $\mu * \sigma$ of μ and σ is defined as follows ;

$$(\mu * \sigma)(x) = \begin{cases} \sup_{x=y*z}\{\min(\mu(y), \sigma(z))\} & \text{where } y, z \in S \\ 0 & \text{if } x \text{ is not expressible as } x = y * z. \end{cases}$$

We extend the concept of Lie algebra to the framework of fuzzy setting and define a fuzzy Lie algebra and discuss their properties and relations among them. A vector space L over a field F with an operation $L \times L \longrightarrow L$, denoted $(x, y) \longrightarrow [xy]$, and called the *bracket* of x and y, is called a *Lie algebra* over F if the following axioms are satisfied ;

(L1) The bracket operation is bilinear.
(L2) $[xx] = 0$, for all $x \in L$.
(L3) $[x[yz]] + [y[zx]] + [z[xy]] = 0$, for all $x, y, z \in L$.

Axiom $(L3)$ is called the *Jacobi identity* [H].

From now on, L is a Lie algebra over a field F.

$(L1)$ and $(L2)$, applied to $[x + y, x + y]$, imply anticommutativity ; $(L2')$ $[xy] = -[yx]$. A subspace K of L is called a *Lie subalgebra* of L if $[xy] \in K$, for all $x, y \in K$.

A subspace I of L is called a *Lie ideal* of L if for all $x, y \in L$ together imply $[xy] \in I$. Since $[xy] = -[yx]$, the condition could just as well be written : $[yx] \in I$.

Definition 1. A fuzzy set μ of L is called a *fuzzy Lie subalgebra* of L if, for all $\alpha \in F$, $x, y \in L$, the following requirements are met ;

(i) $\mu(x + y) \geq \min(\mu(x), \mu(y))$,
(ii) $\mu(\alpha x) \geq \mu(x)$,
(iii) $\mu([xy]) \geq \min(\mu(x), \mu(y))$.

If the condition (iii) is replaced by $\mu([xy]) \geq \max(\mu(x), \mu(y))$, then μ is called a *fuzzy Lie ideal* of L.

Example 2. Let $L = \mathbb{R}^3$ and $[xy] = x \times y$, where $\times$ is cross product, for all $x, y \in L$. Then L is a Lie algebra over a field $\mathbb{R}$ [H].

(1) Define $\mu : \mathbb{R}^3 \longrightarrow [0,1]$ by

$$\mu((x,y,z)) = \begin{cases} 1 & \text{if } x = y = z = 0 \\ \frac{1}{2} & \text{if } x \neq 0,\ y = z = 0 \\ 0 & \text{otherwise,} \end{cases}$$

then μ is a fuzzy Lie subalgebra of L. But μ is not a fuzzy Lie ideal of L, for $\mu([(1,0,0)(1,1,1)]) = \mu((0,1,1)) = 0$, $\max\{\mu((1,0,0)), \mu((1,1,1))\} = \max(\frac{1}{2}, 0) = \frac{1}{2}$.

(2) Define $\sigma : \mathbb{R}^3 \longrightarrow [0,1]$ by

$$\sigma((x,y,z)) = \begin{cases} 1 & \text{if } x = y = z = 0 \\ 0 & \text{otherwise,} \end{cases}$$

then σ is a fuzzy Lie ideal of L.

Lemma 3. *Let μ be a fuzzy Lie subalgebra[resp.fuzzy Lie ideal] of L, then*

(1) $\mu(0) \geq \mu(x)$, *for all* $x \in L$,
(2) $\mu(\alpha x) = \mu(x)$, *for all* $0 \neq \alpha \in F$, *for all* $x \in L$, *in particular,* $\mu(-x) = \mu(x)$
(3) $\mu(x) < \mu(y) \Rightarrow \mu(x - y) = \mu(x) = \mu(y - x)$.
(4) *if* $\mu(x) \neq \mu(y)$, *then* $\mu(x + y) = \min(\mu(x), \mu(y))$.

In [D, R, L1], the following Theorems are the important Theorem, also hold on fuzzified of Lie algebras. The following Theorem 4 will show the relation between fuzzy Lie subalgebras [resp. fuzzy Lie ideals] of L and Lie subalgebras[resp. Lie ideals] of L.

Theorem 4. *A fuzzy set μ of L is a fuzzy Lie subalgebra[resp. fuzzy Lie ideal] of L if and only if the level subsets $\mu_t = \{x \in L : \mu(x) \geq t\}$, for $0 \leq t \leq \mu(0)$, are Lie subalgebras[resp. fuzzy Lie ideals] of L,*

Theorem 5. *Let μ, σ be any two fuzzy Lie ideals of L. Then the sum $\mu + \sigma$ is also fuzzy Lie ideal of L.*

Theorem 6. *Let $\{\mu_i : i \in I\}$ be a set of fuzzy Lie algebras [resp. fuzzy Lie ideals]of L. Then the fuzzy set $(\bigcap_{i \in I} \mu_i)$ of L is also a fuzzy Lie algebra [resp. fuzzy Lie ideal] of L.*

However, union of two fuzzy Lie ideals can not be fuzzified. Let μ, σ be two fuzzy Lie ideals of L. Then $\mu \cup \sigma$ can not be a fuzzy Lie ideal of L. The counterexample is Example 7.

Example 7. Let V be a vector space over a field F. V has basis $(e_1, e_2, \cdots, e_8)$ with Lie brackets are follow ; $[e_1e_2] = e_5$, $[e_1e_3] = e_6$, $[e_1e_4] = e_7$, $[e_1e_5] = -e_8$, $[e_2e_3] = e_8$, $[e_2e_4] = e_6$, $[e_2e_6] = -e_7$, $[e_3e_4] = -e_5$, $[e_3e_5] = -e_7$, $[e_4e_6] = -e_8$, all others $[e_ie_j] = 0, i \leq j$, $[e_ie_j] = -[e_je_i]$.
Then V is a Lie algebra over a field F [J].

Define fuzzy sets μ, σ of V, for all $x \in V$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0,\ e_8 \\ 0.7 & \text{if } x = e_7 \\ 0 & \text{otherwise,} \end{cases}$$

$$\sigma(x) = \begin{cases} 1 & \text{if } x = 0,\ e_7 \\ 0.5 & \text{if } x = e_8 \\ 0 & \text{otherwise.} \end{cases}$$

Hence μ, σ are fuzzy Lie ideals of V.

For, since level Lie ideals, $\mu_1 =< e_8 >, \sigma_1 =< e_7 >$ and $\mu_{0.7} = \sigma_{0.5} =< e_7, e_8 >$ are Lie ideals of V, by Theorem 4. But,

$$\begin{aligned} (\mu \cup \sigma)(e_7 + e_8) &= \max(\mu(e_7 + e_8), \sigma(e_7 + e_8)) \\ &= \max\{\min(\mu(e_7), \mu(e_8)), \min(\sigma(e_7), \sigma(e_8))\} \\ &= \max(0.7,\ 0.5) \\ &= 0.7 \end{aligned}$$

and

$$\begin{aligned} \min((\mu \cup \sigma)(e_7), (\mu \cup \sigma)(e_8)) &= \min\{\max(\mu(e_7), \sigma(e_7)), \max(\mu(e_8), \sigma(e_8))\} \\ &= \min(1, 1) \\ &= 1. \end{aligned}$$

We have

$$\begin{aligned} 0.7 &= (\mu \cup \sigma)(e_7 + e_8) \\ &\geq \min((\mu \cup \sigma)(e_7), (\mu \cup \sigma)(e_8)) \\ &= 1. \end{aligned}$$

This is impossible. Therefore, $\mu \cup \sigma$ is not a fuzzy Lie ideal of V.

Let $(R, +, *)$ be a ring and μ, σ be any two fuzzy ideals of R, then $(\mu * \sigma)$ be a fuzzy ideals of R [DKA2],[K3]. However, Lie bracket of two fuzzy Lie ideals of L can not be fuzzified. Let θ, η be two fuzzy Lie ideals of L. Then $[\theta\eta]$ can not be a fuzzy Lie ideal of L. The counterexample is the Example 8.

Example 8. Let V be the above Example 7.

Define fuzzy sets μ, σ of V, for all $x \in V$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0,\ e_1,\ e_5,\ e_6,\ e_7,\ e_8 \\ 0 & \text{otherwise,} \end{cases}$$

$$\sigma(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.5 & \text{if } x = e_2,\ e_5,\ e_6,\ e_7,\ e_8 \\ 0 & \text{otherwise.} \end{cases}$$

Hence μ, σ are fuzzy Lie ideals of V by Theorem 4, since level Lie ideals $\mu_1 =< e_1, e_5, e_6, e_7, e_8 >$ and $\sigma_{0.5} =< e_2, e_5, e_6, e_7, e_8 >$ are Lie ideals of V.

But, the following condition does not hold

$$[\mu\sigma](e_7 + e_8) \le \min\{[\mu\sigma](e_7), [\mu\sigma](e_8)\}$$

$$(1)\ [\mu\sigma](e_7) = \sup \begin{cases} \min\{\mu(e_1), \sigma(e_4)\} = 0, & e_7 = [e_1 e_4] \\ \min\{\mu(e_2), \sigma(e_6)\} = 0, & e_7 = -[e_2 e_6] \\ \min\{\mu(e_3), \sigma(e_5)\} = 0, & e_7 = -[e_3 e_5] \\ \min\{\mu(e_4), \sigma(e_1)\} = 0, & e_7 = -[e_4 e_1] \\ \min\{\mu(e_6), \sigma(e_2)\} = 0.5, & e_7 = [e_6 e_2] \\ \min\{\mu(e_5), \sigma(e_3)\} = 0, & e_7 = [e_5 e_3] \end{cases} = 0.5.$$

(2) by the same method, $[\mu\sigma](e_8) = 0.5$.

(3)$[\mu\sigma](e_7 + e_8) = \sup\{(i) - (vi)\}$, where

(i) the cases $e_7 + e_8 = [e_1(e_4 - e_5)]$;

$$\begin{aligned} \min\{\mu(e_1), \sigma(e_4 - e_5)\} &= \min\{\mu(e_1), \sigma(e_4), \sigma(e_5)\} \\ &= 0 \text{ since } \sigma(e_4) = 0, \text{ if } e_7 + e_8 = [e_1(e_4 - e_5)] \\ \min\{\mu(e_5 - e_4), \sigma(e_1)\} &= \min\{\mu(e_5), \mu(e_4), \sigma(e_1)\} \\ &= 0 \text{ since } \mu(e_4) = 0, \text{ if } e_7 + e_8 = [(e_5 - e_4)e_1]. \end{aligned}$$

Similarly,

(ii) the cases $e_7 + e_8 = [e_2(e_3 - e_6)]$; then the value is 0.
(iii) the cases $e_7 + e_8 = [e_3(-e_2 - e_5)]$; then the value is 0.
(iv) the cases $e_7 + e_8 = [e_4(-e_1 - e_6)]$; then the value is 0.
(v) the cases $e_7 + e_8 = [e_5(-e_3 - e_1)]$; then the value is 0.
(vi) the cases $e_7 + e_8 = [e_6(-e_2 - e_4)]$; then the value is 0.

If $a, b, c, d, e, a', b', c', d', e' \in F$, then

$$\begin{aligned} e_7 + e_8 &= [(ae_1 + be_5 + ce_6 + de_7)(a'e_2 + b'e_5 + c'e_6 + d'e_7 + e'e_8)] \\ &= aa'e_1e_2 + ab'e_1e_5 + ca'e_6e_2 \\ &= aa'e_5 - ab'e_8 + ca'e_7. \end{aligned}$$

We have $aa' = 0, ab' = -1, ca' = 1$. If $aa' = 0$ then $a = 0$ or $a' = 0$, since F is field. Therefore, $ab' = -1$ and $ca' = 1$ are impossible. Similarly, the cases

$e_7 + e_8 = [(ae_2 + be_5 + ce_6 + de_7 + ee_8)(a'e_1 + b'e_5 + c'e_6 + d'e_7)]$ is impossible. Hence $[\mu\sigma](e_7 + e_8) = 0$.

We have

$$0 = [\mu\sigma](e_7 + e_8) \geq \min\{[\mu\sigma](e_7), [\mu\sigma](e_8)\} = 0.5.$$

This is impossible. Therefore, $[\mu\sigma]$ is not a fuzzy Lie ideal of V.

Question. Let $\mathcal{F}(L)$ denote the set of all fuzzy Lie ideals of L. Can we give a Lie algebra structure to $\mathcal{F}(L)$?

Kumar introduced the irreducible fuzzy ideals of a ring R [K2]. Using the same method, we can be fuzzified on a Lie ideals of L. A Lie ideal I of L is said to be *irreducible* if for two Lie ideals A, B of L, $A \cap B = I$ implies $A = I$ or $B = I$ [H].

Definition 9. A fuzzy Lie ideal μ of L is said to be *irreducible* if for any two fuzzy Lie ideals σ, ν of L,

$$\sigma \cap \nu = \mu \text{ implies } \sigma = \mu \text{ or } \nu = \mu.$$

Example 10. (1) In Example 2(2), σ is irreducible.

(2) Let $G = Fx_1 \oplus \cdots Fx_n \oplus Fy_1 \oplus \cdots Fy_n \oplus Fz$ be a Heisenberg Lie algebra with $[x_i y_i] = z$, otherwise Lie bracket$= 0$ [H]. A fuzzy set μ is defined, for all $a \in G$ by

$$\mu(x) = \begin{cases} 1 & \text{if } a = 0 \\ 0.5 & \text{if } a = z \\ 0 & \text{otherwise} \end{cases}$$

Then μ is a fuzzy Lie ideal of G. For, since a level Lie ideal $\mu_{0.5} =< z >$ is a Lie ideal of G, by Theorem 4.

But, μ is not irreducible (The proof is next Theorem 11(2)).

(3) In Example 7, define fuzzy sets α, β, γ of V, for all $x \in V$ by

$$\alpha(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta(x) = \begin{cases} 1 & \text{if } x = 0, e_7, e_8 \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.8 & \text{if } x = e_8 \\ 0.7 & \text{if } x = e_7 \\ 0.6 & \text{if } x = e_6 \\ 0.5 & \text{if } x = e_5 \\ 0 & \text{otherwise.} \end{cases}$$

Hence α, β, γ are fuzzy Lie ideals of V. For, since level Lie ideals, $\gamma_{0.8} =< e_8 >$, $\beta_1 = \gamma_{0.7} =< e_7, e_8 >$, $\gamma_{0.6} =< e_6, e_7, e_8 >$ and $\gamma_{0.5} =< e_5, e_6, e_7, e_8 >$ are Lie ideals of V, by Theorem 4.

Then α is irreducible, but β, γ are not irreducible (The proof is next Theorem 11(2) and Remark 12).

Theorem 11. *Let μ be a nonconstant irreducible fuzzy Lie ideal of L. Then the following statements hold ;*

(1) $1 \in \mu(L)$.
(2) $|\mu(L)| = 2$, *where* $|\mu(L)|$ *is the cardinality of* $\mu(L)$.
(3) $\mu_1 = \{x \in L : \mu(x) = \mu(0) = 1\}$ *is an irreducible Lie ideal of* L.

Remark 12. In Theorem 11, the converse of (2) does not hold. In Example 7 and 10(3), $|\beta(V)| = 2$, but β is not irreducible. Define fuzzy sets τ, ν of V, for all $x \in V$ by

$$\tau(x) = \begin{cases} 1 & \text{if } x = 0, e_6, e_7, e_8 \\ 0 & \text{otherwise,} \end{cases}$$
$$\nu(x) = \begin{cases} 1 & \text{if } x = 0, e_5, e_7, e_8 \\ 0 & \text{otherwise.} \end{cases}$$

Hence τ, ν are fuzzy Lie ideals of V. For, since level Lie ideals, $\tau_1 =< e_6, e_7, e_8 >$ and $\nu_1 =< e_5, e_7, e_8 >$ are Lie ideals of V, by Theorem 4.

We have $\beta \neq \tau$ and $\beta \neq \nu$. However, $\beta = \tau \cap \nu$. For

$$(\tau \cap \nu)(e_6) = \min(\tau(e_6), \nu(e_6)) = \min(1, 0) = 0 = \beta(e_6),$$

$$(\tau \cap \nu)(e_5) = \min(\tau(e_5), \nu(e_5)) = \min(0, 1) = 0 = \beta(e_5),$$

and others are trivial.

Hence β is not irreducible.

Kumar reflects the effect of a homomorphism on the sum and intersection of fuzzy Lie ideals of a ring R [K2]. Using the same method, we can be fuzzified on a Lie ideals of L, and reflects the effect of a homomorphism on the sum and intersection of fuzzy Lie ideals of L. Let L, L' be two Lie algebras over a field F. A linear transformation $\phi : L \longrightarrow L'$ is called a *Lie homomorphism* if $\phi([xy]) = [\phi(x)\phi(y)]$, for all $x, y \in L$ [H].

Proposition 13. *Let $\phi : L \longrightarrow L'$ be a Lie homomorphism. If μ is a fuzzy Lie algebra [resp. fuzzy Lie ideal] of L', then the fuzzy set $\phi^{-1}(\mu)$ of L is also a fuzzy Lie algebra [resp. fuzzy Lie ideal].*

Proposition 14. *Let $\phi : L \longrightarrow L'$ be a Lie homomorphism. If μ is a fuzzy Lie subalgebra of L, then the fuzzy set $\phi(\mu)$ of L' is also a fuzzy Lie subalgebra of L'.*

Proposition 15. *Let $\phi : L \longrightarrow L'$ be a surjective Lie homomorphism. If μ is a fuzzy Lie ideal of L, then the fuzzy set $\phi(\mu)$ of L' is also a fuzzy Lie ideal of L'.*

Theorem 17. *Let $\phi : L \longrightarrow L'$ be a surjective Lie Homomorphism. Then*

(1) *if μ, σ are two fuzzy Lie ideals of L, then $\phi(\mu + \sigma) = \phi(\mu) + \phi(\sigma)$*

(2) *if $\{\mu_i : i \in I\}$ is a set of $\phi-$invariant fuzzy Lie ideal of L, then*

$$\phi(\bigcap_{i\in I} \mu_i) = \bigcap_{i\in I} \phi(\mu_i).$$

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