LATTICES OF FUZZY SUBGROUPOIDS, FUZZY SUBMONOIDS AND FUZZY SUBGROUPS

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Abstract: We redefine the sup-min product of fuzzy subsets and discuss the redefined sup-min products of fuzzy subgroupoids, fuzzy submonoids and fuzzy subgroups. And we study lattice structures of the lattices of fuzzy subgroupoids, fuzzy submonoids and fuzzy subgroups.

1. Introduction. The sup-min product of two fuzzy subsets was defined by Wu [10] and Liu [7] and has been studied by many researchers. In particular, the sup-min product of two normal fuzzy subgroups is again a normal fuzzy subgroup and can be applied to obtain an explicit formula of the join of two normal fuzzy subgroups. Wu [10] initiated lattice theoretic studies in fuzzy group theory. Then Ajmal and Thomas in [2] introduced various types of sublattices such as $\mathcal{L}_t, \mathcal{L}_f, \mathcal{L}_{ft}$, and $\mathcal{L}_{fn_t}$ of the lattice $\mathcal{L}$ of all fuzzy subgroups of a group G (for details see [2]) and proved that $\mathcal{L}_{fn_t}$ is a modular sublattice of $\mathcal{L}$. After that Gupta and Ray [5] studied the structure of the lattice $FN(G)$ of all normal fuzzy subgroups of G in connection with the sup-min product and established the modularity of the lattice $FN(G)$ which is an extension of Wu's result [10] that the set of all normal fuzzy subgroups A of G such that $A(e) = 1$ constitutes a modular lattice. Since the formula $A \cup B \cup (A \circ B)$ of the join of A and B in the lattice $FN(G)$ given by Gupta and Ray [5] is not valid in the lattice $FG(G)$ of all fuzzy subgroups of G, the structure of the lattice $FG(G)$ could not be studied and it has not been known to the authors in [5] whether the lattice $FN(G)$ is a sublattice of the lattice $FG(G)$ or not. In a recent paper, Ajmal [1] has shown that $FN(G)$ is indeed a sublattice of the lattice $FG(G)$. In the same paper, a direct proof of modularity of the lattice $FN(G)$ is provided; also this proof does not require the construction of the fuzzy set $A \cup B \cup (A \circ B)$. Moreover, the sup-min product of two fuzzy subgroups is generally not a fuzzy subgroupoid at all.

In this paper we redefine the sup-min product of two fuzzy subsets and show that the problems above can be overcome by the redefined sup-min product. Actually, the redefined sup-min products of two fuzzy subgroupoids, two fuzzy submonoids and two fuzzy subgroups is a fuzzy subgroup, a fuzzy submonoid and a fuzzy

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subgroup, respectively. We discuss structures of the set $FI(S)$ of all fuzzy subgroupoids of a semigroup S, the set $FD(M)$ of all fuzzy submonoids of a monoid M, and the set $FG(G)$ of a group G under redefined sup-min product and give explicit formulas of the joins of A and B in the lattices $FI(S)$, $FD(M)$, and $FG(G)$, respectively. And we show that, in the sequence $FI(G) \supseteq FD(G) \supseteq FG(G) \supseteq FN(G) \supseteq FC(G)$ of lattices where G is a given group and $FC(G)$ denotes the lattice of all commutative fuzzy subgroups of G, each lattice is a sublattice of the former lattice.

2. Redefined sup-min product of two fuzzy subsets. We now redefine the sup-min product of two fuzzy subsets. The redefined product $\circledcirc$ is a modification of the sup-min product $\circ$ of two fuzzy subsets.

DEFINITION 2.1. *Let A and B be fuzzy subsets of a semigroup G. Then the product $A \circledcirc B$ of A and B is the fuzzy subset of G defined by*

$$(A \circledcirc B)(x) = \sup_{\substack{x=a_1b_1\cdots a_nb_n \\ n\in\mathbb{N}}} \min\{A(a_1), B(b_1), \cdots, A(a_n), B(b_n)\}.$$

THEOREM 2.2. *Let A and B be fuzzy subsets of a semigroup G. Then $A \circledcirc B$ is a fuzzy subgroupoid of G.*

Generally the product $A \circ B$ of two fuzzy subsets A and B is not a fuzzy subgroupoid even if A and B are fuzzy subgroups. So Theorem 2.2 says that the redefined product is rather useful in the study of fuzzy subgroupoids, fuzzy submonoids, and fuzzy subgroups. In fact, it will be later shown that the redefined product of any two fuzzy submonoids [resp. any two fuzzy subgroups] is a fuzzy submonoid [resp. a fuzzy subgroup] and can be applied to obtain a formula of the join of two fuzzy submonoids [resp. two fuzzy subgroups]. And the following theorem substantiates that.

THEOREM 2.3. *Let A and B be fuzzy subsets of a semigroup G. Then $A \circ B$ is a fuzzy subgroupoid of G if and only if $A \circ B = A \circledcirc B$.*

THEOREM 2.4. *Let $FE(G)$ be the set of all fuzzy subsets A of a monoid G such that $A(e) \geq A(x)$ for all $x \in G$. Then $FE(G)$ is a commutative semigroup under product $\circledcirc$.*

THEOREM 2.5. *Let $FD(G)$ be the set of all fuzzy submonoids of a monoid G. Then $FD(G)$ is a subsemigroup of the semigroup $FE(G)$ in Theorem 2.4 with identity and $A \circledcirc A = A$ for every $A \in FD(G)$, i.e., $FD(G)$ is a commutative idempotent monoid under product $\circledcirc$.*

THEOREM 2.6. *Let $FG(G)$ be the set of all fuzzy subgroups of a group G. Then $FG(G)$ is a submonoid of the monoid $FD(G)$ of all fuzzy submonoids of G, i.e., $FG(G)$ is a commutative idempotent monoid under product $\odot$.*

THEOREM 2.7. *Let $FG(G)$ [resp. $FN(G), FC(G)$] be the set of all fuzzy subgroups [resp. all normal fuzzy subgroups, all commutative fuzzy subgroups] of a group G. Then $FN(G)$ is a submonoid of the monoid $FG(G)$ and $FC(G)$ is a submonoid of $FN(G)$.*

3. Lattices of fuzzy subgroupoids, fuzzy submonoids and fuzzy subgroups. In this section, we deal with lattices of fuzzy subgroupoids, fuzzy submonoids and fuzzy subgroups, and so we deal with the concepts of generated fuzzy subgroupoids, generated fuzzy submonoids and generated fuzzy subgroups. Generating techniques are very important in classical algebra as well as in fuzzy algebraic structures. Therefore we give a brief review of the developement of this concept in fuzzy framework. It was none else but Rosenfeld who introduced the idea of a generated fuzzy subgroupoid in his celebrated paper [9]. In 1984, Biacino and Gerla [3] introduced closure systems of L-subalgebras and provided formulas for generated L-subalgebras, L-subgroups and normal L-subgroups of required type by a given fuzzy set. Then in 1990, Dixit et al. [4], probably unaware of the work of Biacino and Gerla, provided specific construction of the fuzzy subgroup generated by a given fuzzy subset of finite range. Later on Kumar [6] in 1992 gave the most general construction including a mistake which can be trivially removed. A more recent and notable attempt in this direction is due to Ray [8]. There is a striking resemblance between Ray's results and Biacino and Gerla's results, although Biacino and Gerla's results were obtained much earlier.

THEOREM 3.1. *Let $FI(G)$ be the set of all fuzzy subgroupoids of a semigroup G. Then $FI(G)$ is a lattice under meet $A \wedge B$ and join $A \vee B$ given by $A \wedge B = A \cap B$ and $A \vee B = J_{A,B}$ where $J_{A,B}$ is the fuzzy subset of G defined by*

$$J_{A,B}(x) = \sup_{\substack{x = x_1 \cdots x_n \\ n \in \mathbb{N} \\ C_i \in \{A,B\}}} \min\{C_1(x_1), \cdots, C_n(x_n)\}.$$

THEOREM 3.2. *The set $FD(G)$ of all fuzzy submonoids of a monoid G is a sublattice of the lattice $FI(G)$ of all fuzzy subgroupoids of G.*

THEOREM 3.3. *Let A and B be fuzzy submonoids of a monoid G. Then $A \cup B \cup (A \circledcirc B)$ is the join $A \vee B$ of A and B in the lattice $FD(G)$ of all fuzzy submonoids of G.*

Note that the lattice $FD(G)$ of all fuzzy submonoids of a monoid G is not a sublattice of the lattice $FE(G)$ of all fuzzy subsets A of G such that $A(e) \geq A(x)$ for all $x \in G$.

THEOREM 3.4. *The set $FG(G)$ of all fuzzy subgroups of a group G is a sublattice of the lattice $FD(G)$ of all fuzzy submonoids of G.*

THEOREM 3.5. *Let $FG(G)$ [resp. $FN(G), FC(G)$] be the set of all fuzzy subgroups [resp. all normal fuzzy subgroups, all commutative fuzzy subgroups] of a group G. Then $FN(G)$ is a sublattice of the lattice $FG(G)$ and $FC(G)$ is a sublattice of $FN(G)$.*

REMARK 3.6. *In a recent paper, the first half of Theorem 3.5 has been proved by Ajmal [2]. Indeed, he has shown, using the concept of strong level subsets, that $FN(G)$ is a modular sublattice of $FG(G)$ without any formula of joins.*

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