P_L-LIMIT STRUCTURES

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ABSTRACT

We define a P_L-limit structure and a P_L-Cauchy structure and obtain a completion of a separated P_L-Cauchy structure.

1. Intriduction

In 1992, P. Eklund and W. Gähler([3]) introduced the notions of generalized limit and Cauchy structure. Each of these structures is related to a certain covariant functor $\Phi = (\phi, \leq)$ from the category **Set** of sets to the category **SLat** of meet-semilattices satisfying some conditions on products.

Let Φ : **Set** $\rightarrow$ **SLat** be a covariant functor and ϕ : **Set** $\rightarrow$ **Set** the underlying set functor of Φ.

Let X and Y be sets. For $\mathcal{M} \in \phi X$ and $\mathcal{N} \in \phi Y$, the Φ-product $\mathcal{M} \times \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ is the smallest element $\mathcal{L}$ of $\phi(X \times Y)$, for which $\phi\pi_1(\mathcal{L}) = \mathcal{M}$ and $\phi\pi_2(\mathcal{L}) = \mathcal{N}$, provided it exists, where π_1 and π_2 are projections of $X \times Y$.

Consider the following conditions on the functor $\Phi = (\phi, \leq)$.

(P) For every $\mathcal{M} \in \phi X$ and $\mathcal{N} \in \phi Y$, the Φ-product of $\mathcal{M}$ and $\mathcal{N}$ exists.

(Pr) If $f : X \rightarrow U$ and $g : Y \rightarrow V$ are maps, then $\phi(f \times g)(\mathcal{M} \times \mathcal{N}) = \phi f(\mathcal{M}) \times \phi g(\mathcal{N})$ for every $\mathcal{M} \in \phi X$ and $\mathcal{N} \in \phi U$.

(D) If $\mathcal{M} \in \phi X$ and $\mathcal{N}_1, \mathcal{N}_2 \in \phi Y$, then $\mathcal{M} \times (\mathcal{N}_1 \wedge \mathcal{N}_2) = (\mathcal{M} \times \mathcal{N}_1) \wedge (\mathcal{M} \times \mathcal{N}_2)$.

We say that ϕ is connected if $\phi 1$ is a singleton, where $1 = \{p\}$ is a singleton set. If ϕ is connected, then a natural transformation $\eta : id \rightarrow \phi$ is uniquely defined. In this section we assume that the functor $\Phi = (\phi, \leq)$ fulfills the conditions (P), (Pr) and (D) and ϕ is connected.

Definition 1.1. Let X be a set. A subset t of $\phi X \times X$ is called a Φ-*limit structure* if the following conditions are satisfied, where $\mathcal{M} \underset{t}{\to} x$ means $(\mathcal{M}, x) \in t$.

(L1) $\eta_X(x) \underset{t}{\to} x$ for all $x \in X$.

(L2) $\mathcal{M} \underset{t}{\to} x$ and $\mathcal{M} \leq \mathcal{N}$ imply $\mathcal{N} \underset{t}{\to} x$.

(L3) If $\mathcal{M} \underset{t}{\to} x$ and $\mathcal{N} \underset{t}{\to} x$, then $\mathcal{M} \wedge \mathcal{N} \underset{t}{\to} x$

The pair (X, t) is called a Φ-*limit* space.

Definition 1.2. A map $f : (X, t) \to (Y, u)$ between Φ-limit spaces is said to be Φ-*continuous* if $\mathcal{M} \underset{t}{\to} x$ implies $\phi f(\mathcal{M}) \underset{u}{\to} f(x)$.

Let **Φ-Lim** be the category of all Φ-limit spaces and Φ-continuous maps between them.

Definition 1.3. A Φ-*Cauchy structure* on a set X is a subset s of ϕX satisfying the following conditions.

(C1) $\eta_X(x) \in s$ for all $x \in X$.

(C2) $\mathcal{M} \in s$ and $\mathcal{M} \leq \mathcal{N}$ imply $\mathcal{N} \in s$.

(C3) If $\mathcal{M}$, $\mathcal{N} \in s$ and $\{\mathcal{M}, \mathcal{N}\}$ has an upper bound in $(\phi X, \leq)$, then $\mathcal{M} \wedge \mathcal{N} \in s$.

The pair (X, s) is called a Φ-*Cauchy space*.

Definition 1.4. A map $f : (X, s) \to (Y, u)$ between Φ- Cauchy spaces is said to be Φ-*Cauchy continuous* if $\mathcal{M} \in s$ implies $\phi f(\mathcal{M}) \in u$.

Let **Φ-Chy** be the category of all Φ-Cauchy spaces and Φ-Cauchy continuous maps between them.

Let (X, t) and (Y, u) be Φ-limit spaces and let $C(X, Y)$ denote the set of all Φ-continuous maps from (X, t) to (Y, u).

Proposition 1.5. *The subset c of $\phi C(X, Y) \times C(X, Y)$ defined by*

$$\mathcal{K} \underset{c}{\to} f \quad \textit{if and only if} \quad \phi ev(\mathcal{M} \times \mathcal{K}) \underset{u}{\to} f(x)$$
$$\textit{for each } x \in X \textit{ and } \mathcal{M} \underset{t}{\to} x,$$

where $ev : X \times C(X, Y) \to Y$ is the evaluation map $(x, f) \longmapsto f(x)$, is a Φ-limit structure on $C(X, Y)$.

Theorem 1.6. **Φ-Lim** *is a cartesian closed topological category.*

Let (X, s) and (Y, u) be Φ-Cauchy spaces and let $C_c(X, Y)$ be the set of all Φ-Cauchy continuous maps between them.

Proposition 1.7. *The subset d of $\phi C_c(X,Y)$, defined by*

$\mathcal{K} \in d$ *if and only if* $\phi ev(\mathcal{M} \times \mathcal{K}) \in u$ *for each* $\mathcal{M} \in s$,

is a Φ*-Cauchy structure on* $C_c(X,Y)$.

Theorem 1.8. Φ**-Chy** *is a cartesian closed topological category.*

2. P_L-limit spaces and P_L-Cauchy spaces

Let L be a linear complete Heyting algebra. A subset $\mathcal{F}$ of L^X, the set of all maps from X to L, is called an *L-prefilter* on X if it satisfies the following:

(LP1) $\underline{\alpha} \in \mathcal{F}$ for all $\alpha > 0$ and $\underline{0_L} \notin \mathcal{F}$, where $\underline{\alpha}$ is the constant map with value α,

(LP2) if $A \in \mathcal{F}$ and $A \leq B$ then $B \in \mathcal{F}$,

(LP3) for all $A,\ B \in \mathcal{F}$, we have $A \wedge B \in \mathcal{F}$.

$A \leq B$ and $A \wedge B$ are defined pointwise.

A subset $\mathcal{B}$ of an L-prefilter $\mathcal{F}$ on X is called a *basis* for $\mathcal{F}$ if for each $A \in \mathcal{F}$ there exists $B \in \mathcal{B}$ such that $B \leq A$. For a given set X a subset $\mathcal{B}$ of L^X is a basis for an L-prefilter on X if it satisfies the following:

(LPB1) $\underline{\alpha} \in \mathcal{B}$ for all $\alpha > 0$ and $\underline{0_L} \notin \mathcal{B}$,

(LPB2) for all $A, B \in \mathcal{B}$, there exists $C \in \mathcal{B}$ such that $C \leq A \wedge B$.

The *L-prefilter $\mathcal{F}$ generated by $\mathcal{B}$* is defined by $\{A \in L^X : B \leq A \text{ for some } B \in \mathcal{B}\}$ and is denoted by $\langle \mathcal{B} \rangle$. For a map $f : X \to Y$ and a prefilter (base) $\mathcal{B}$ on X, $\{B \in L^Y : f(A) \leq B \text{ for some } A \in \mathcal{B}\}$ is an L-prefilter on Y and is denoted by $f(\mathcal{B})$. In particular, if $\mathcal{F} = \langle \mathcal{B} \rangle$, then $f(\mathcal{F}) = f(\mathcal{B})$.

For a set X, let $\phi(X) = P_L(X)$ be the set of all L-prefilters on X. Then $(P_L(X), \subseteq)$ is a meet-semilattice with $\mathcal{F} \cap \mathcal{G} = \{A \vee B : A \in \mathcal{F} \text{ and } B \in \mathcal{G}\}$ for all $\mathcal{F}, \mathcal{G} \in P_L(X)$. So we can define a functor $P_L :$ **Set** $\to$ **SLat**. And since $P_L 1$ is singleton $\{\{\underline{\alpha} : \alpha > 0_L\}\}$, there exists a unique natural transformation $\eta : id \to P_L$ defined by $\eta_X(x) = \{A \in L^X : \mu_A(x) > 0_L\} = \langle x \rangle$.

Let X and Y be sets. For $\mathcal{F} \in P_L(X)$ and $\mathcal{G} \in P_L(Y)$, $\langle \{A \circ \pi_1 \wedge A \circ \pi_2 : A \in \mathcal{F}, B \in \mathcal{G}\} \rangle$ is the smallest element in $P_L(X \times Y)$ of which first and second projections are $\mathcal{F}$ and $\mathcal{G}$, respectively.

Proposition 2.1. $(P_L, \subseteq)$ *fulfills* (P), (Pr) and (D).

So we can define a P_L-limit structure and a P_L-Cauchy structure replacing Φ with P_L in 1.1 and 1.3, respectively.

A *P_L-limit structure* t is a subset of $\mathcal{F}_L(X) \times X$, subject to the following axioms: where $\mathcal{F} \underset{t}{\to} x$ means $(\mathcal{F}, x) \in t$,

$(P_L\mathrm{L}1)$ $\langle x \rangle = \{A \in L^X : \mu_A(x) > 0_L\} \underset{t}{\to} x$ for all $x \in X$,

$(P_L\mathrm{L}2)$ $\mathcal{F} \underset{t}{\to} x$ and $\mathcal{F} \subseteq \mathcal{G}$ imply $\mathcal{G} \underset{t}{\to} x$,

$(P_L\mathrm{L}3)$ if $\mathcal{F} \underset{t}{\to} x$ and $\mathcal{G} \underset{t}{\to} x$, then $\mathcal{F} \cap \mathcal{G} \underset{t}{\to} x$.

The pair (X, t) is called a *P_L-limit space*. A map $f : (X, t) \to (Y, u)$ between P_L-limit spaces is said to be *P_L-continuous* if $\mathcal{F} \underset{t}{\to} x$ implies $f(\mathcal{F}) \underset{u}{\to} f(x)$.

Let P_L-**Lim** denote the category of all P_L-limit spaces and all P_L-continuous maps between them.

Proposition 2.2. *P_L-**Lim** is a cartesian closed topological category.*

Theorem 2.3. *P_L-**Lim** is a quasitopos, i.e. final epi-sinks in P_L-**Lim** are preserved by pullbacks.*

A *P_L-Cauchy structure* s on X is a subset of $\mathcal{F}_L(X)$, subject to the following axioms:

$(P_L\mathrm{C}1)$ $\langle x \rangle \in s$ for all $x \in X$,

$(P_L\mathrm{C}2)$ $\mathcal{F} \in s$ and $\mathcal{F} \subseteq \mathcal{G}$ imply $\mathcal{G} \in s$,

$(P_L\mathrm{C}3)$ if $\mathcal{F}, \mathcal{G} \in s$ and $\{\mathcal{F}, \mathcal{G}\}$ has an upper bound in $(\mathcal{F}(X), \subseteq)$, i.e. $A \cap B \neq \underline{0}_L$ for all $A \in \mathcal{F}$, $B \in \mathcal{G}$, then $\mathcal{F} \cap \mathcal{G} \in s$.

The pair (X, s) is called a *P_L-Cauchy space*. A map $f : (X, s) \to (Y, u)$ between P_L-Cauchy spaces is said to be *P_L-Cauchy continuous* if $\mathcal{F} \in s$ implies $f(\mathcal{F}) \in u$.

Let P_L-**Chy** denote the category of all P_L-Cauchy spaces and P_L-Cauchy continuous maps between them.

Proposition 2.4. *P_L-**Chy** is a cartesian closed topological category.*

Remark. *For a P_L-Cauchy space (X, s), the structure t_s on X, defined by $\mathcal{F} \underset{t_s}{\longrightarrow} x$ if and only if $\mathcal{F} \cap \langle x \rangle \in s$, is a P_L-limit structure on X and P_L-Cauchy continuity implies P_L-continuity with respect to the associated P_L-limit structures.*

Definition 2.5. (1) A P_L-Cauchy space (X, s) is said to be *complete* if for each Φ-Cauchy object $\mathcal{M}$ there exists $x \in X$ such that $\mathcal{F} \underset{t_s}{\longrightarrow} x$.

(2) A P_L-Cauchy space (X, s) is said to be *separated* if $\mathcal{F} \underset{t_s}{\longrightarrow} x, y$ implies $x = y$.

We obtain P. Eklund and W. Gähler completion([1]) of separated P_L-Cauchy spaces.

For a separated P_L-Cauchy space (X, s), a relation $\sim$ on s defined by $\mathcal{F} \sim \mathcal{G}$ if and only if $\mathcal{F} \cap \mathcal{G} \in s$ for $\mathcal{F}, \mathcal{G} \in s$ is an eqivalence relation on s. Let $X^{\sim} = \{\mathcal{F}^{\sim} : \mathcal{F} \in s\}$, where $\mathcal{F}^{\sim} = \{\mathcal{G} \in s : \mathcal{F} \sim \mathcal{G}\}$ is the equivalence class of $\mathcal{F}$ with respect to $\sim$ and let $\iota : X \to X^{\sim}$ be a map $x \longmapsto \langle x \rangle^{\sim}$ and let

$$s^{\sim} = \{\mathcal{K} \in P_L(X^{\sim}) : \iota(\mathcal{F}) \cap \langle \mathcal{F}^{\sim} \rangle \subseteq \mathcal{K} \text{ for some } \mathcal{F} \in s\}.$$

Then $(X^{\sim}, s^{\sim})$ is a separated and complete P_L-Cauchy space and s coincides with the initial P_L-Cauchy structure on X with respect to ι.

We call $(X^{\sim}, s^{\sim})$ the *completion* of (X, s).

Theorem 2.6. *For each seperated P_L-Cauchy space (X, s), $(X^{\sim}, s^{\sim})$ is an epireflection of (X, s) with respect to P_L-$\mathbf{Chy}_{\mathrm{sep,cpl}}$ and hence P_L-$\mathbf{Chy}_{\mathrm{sep,cpl}}$ is an epireflective subcategory of P_L-$\mathbf{Chy}_{\mathrm{sep}}$.*

3. References

[1] P. Eklund and W. Gähler, *Generalized Cauchy Spaces*, Math. Nachr. 147 (1990) 219-233.

[2] P. Eklund and W. Gähler, *Fuzzy Filter Functors and Convergence*, in: Applications of Category Theory to Fuzzy Subsets, Kluwer Academic Publishers, Dordrecht/ Boston/ London (1992), 109-136.

[3] P.Eklund and W. Gähler, *Contributions to Fuzzy Convergence*, in: Recent Developments of General Topology and its Applications, international Conference in Memory of Felix Hausdorff Math. Reserch 67, Berlin (1992) 118-123.