A Novel Fuzzy Morphology, Part I: Definitions

Yonggwan Won
Electronics and Telecommunications Research Institute
Taejon-City, Yoosung-Ku, Kajung-Dong 161

Bae-Ho Lee
Department of Computer Engineering
Chonnam National University
Kwangju-City, Puk-Ku, Yongbong-Dong 300

ABSTRACT

A novel definition for fuzzy mathematical morphology is described. The generalized-mean operator plays the key role for this definition. Several hard constraints for standard generalized-mean have been eliminated. Complete mathematical description for obtaining fuzzy erosion and dilation is provided. The definitions are well suited for neural network implementation. Therefore, the parameters for the fuzzy definition can be optimized using neural network learning paradigm.

I. INTRODUCTION

Mathematical morphology has been employed in various image processing tasks such as edge detection, noise suppression, region filling, skeletonization, smoothing, segmentation, texture analysis, thinning, and more. It has also been used in pattern recognition as a feature extraction methodology. The theoretical foundations of mathematical morphology lie in *set theory* which is well-suited for binary images [Matheron, Serra 82a]. Binary morphology has been extended to gray-scale morphology using umbra techniques [Nakagawa, Serra 82a, Sternberg, Giardina, Haralick, Maragos 87] and the lattices theory [Serra 82b, Heijmans]. Fundamental operations for mathematical morphology are *erosion* and *dilation* which use the ordinary *min* and *max* operators.

Fuzzy set theory [Zadeh] has been successfully used to describe phenomena and systems with imprecise linguistic terms (i.e., tall, very tall) used in everyday. A system can be either crisp or fuzzy, depending on how the membership for an item is determined. While crisp (Boolean) systems allow the membership to be either one or zero, fuzzy systems allow for the degree of membership being a number in the range of [0, 1]. In general, for objective function based algorithms which iteratively minimize a criterion function until a global or local minimum is reached, the performance of fuzzy versions is superior to that of the corresponding crisp version [Bezdeck]. Furthermore, fuzzy algorithms are less probable to be trapped in local minima [Bezdeck].

Main purpose of this paper is to introduce a new definition of fuzzy morphology. In Section II, we first review the standard morphological operations (erosion and dilation) and the generalized-mean operator [Dujmovic, Dyckhoff] and its properties. In Section III, we then provide complete mathematical description for obtaining the definitions of fuzzy morphological operations using generalized-mean operator. Finally, in Section IV, conclusions and further works are discussed.

II. BACKGROUNDS

In this section, we first briefly describe the standard definition of two essential operations; gray-scale erosion and dilation. We then introduce generalized-mean operator which plays key roles in defining our novel fuzzy morphological erosion and dilation.

2.1. GRAY-SCALE EROSION AND DILATION

Erosion and dilation are fundamental operations for mathematical morphology. The theoretical foundations of mathematical morphology lie in *set theory* which is well-suited for binary images [Matheron, Serra 82a]. An extension of binary operations to gray-scale operations can be achieved by two different ways: umbra transform [Nakagawa, Serra 82a, Sternberg, Giardina, Haralick, Maragos 87] and lattice theory [Serra 82b, Heijmans]. We briefly describe the definition of those operations, and details of theory, other operations, properties and applications are widely available from the morphology literature [Dougherty, Haralick, Maragos 87, Maragos 89, Matheron, Serra 82ab, Sternberg].

The *erosion* of a function f by a structuring element g is defined by

$$(f \ominus g)(\mathbf{x}) = \max \{y : g_{\mathbf{x}} + y << f\}. \qquad (1)$$

The erosion at a point $\mathbf{x}$ can be done by two steps: (1) move the structuring element spatially so that its origin (origin of Euclidean space) is located at $\mathbf{x}$, and (2) find the maximum amount we can offset (push-up) the structuring element while it is beneath the signal. Obviously, $D[g_{\mathbf{x}}] \subseteq D[f]$ in order to satisfy "beneath" condition, where D indicates the domain. An examples of gray-scale erosion is shown in Fig. 1.

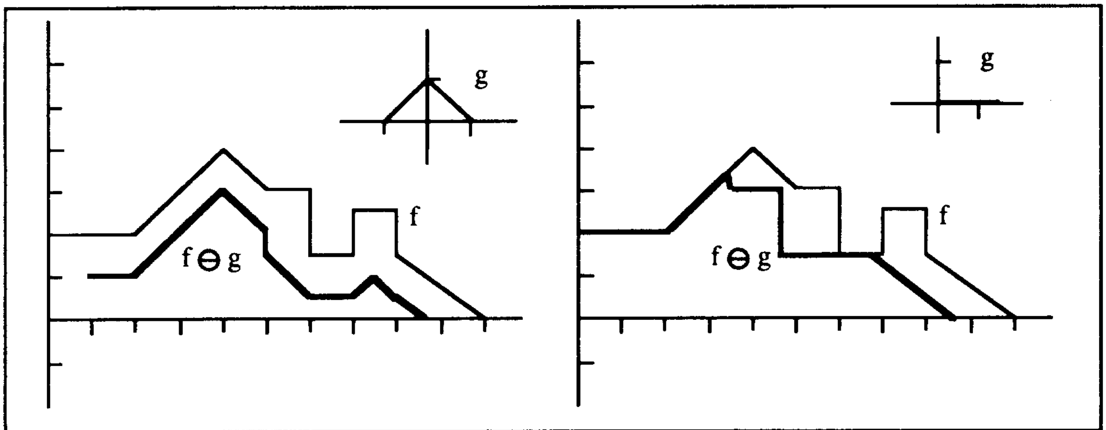

Fig. 1. An examples of gray-scale erosion.

Instead of finding maximum "offset" at a point $\mathbf{x}$, we can find the "minimum difference" between $f(\mathbf{z})$'s and $g_{\mathbf{x}}(\mathbf{z})$'s for all $\mathbf{z} \in D[g_{\mathbf{x}}]$. This notion leads to the formulation of erosion

$$(f \ominus g)(\mathbf{x}) = \min \{f(\mathbf{z}) - g_{\mathbf{x}}(\mathbf{z}) : \mathbf{z} \in D[g_{\mathbf{x}}]\}. \qquad (2)$$

Note that $(f \ominus g)(\mathbf{x})$ is only defined at any point where $g_{\mathbf{x}} << f$.

Gray-scale *dilation* can be defined in a dual manner to gray-scale erosion. Before giving the definitions, we motivate the duality principle by showing how dilation can be viewed as an erosion. Instead of translating the structuring element and finding the maximum offset while keeping the structuring element beneath the signal, we can (i) take the "reflection" of the structuring element g, (ii) move the reflected structuring element g^* to a point $\mathbf{x}$, and (iii) find the "minimum" offset for the reflected-translated structuring element $(g^*)_\mathbf{x}$ to be "above" the signal. We should note that the signal is restricted to the domain of the reflected-translated structuring element $(g^*)_\mathbf{x}$. Figure 2 illustrates an example of gray-scale dilation which is formalized mathematically by

$$(f \oplus g)(\mathbf{x}) = \min \{y : (g^*)_\mathbf{x} + y >> f\}. \tag{3}$$

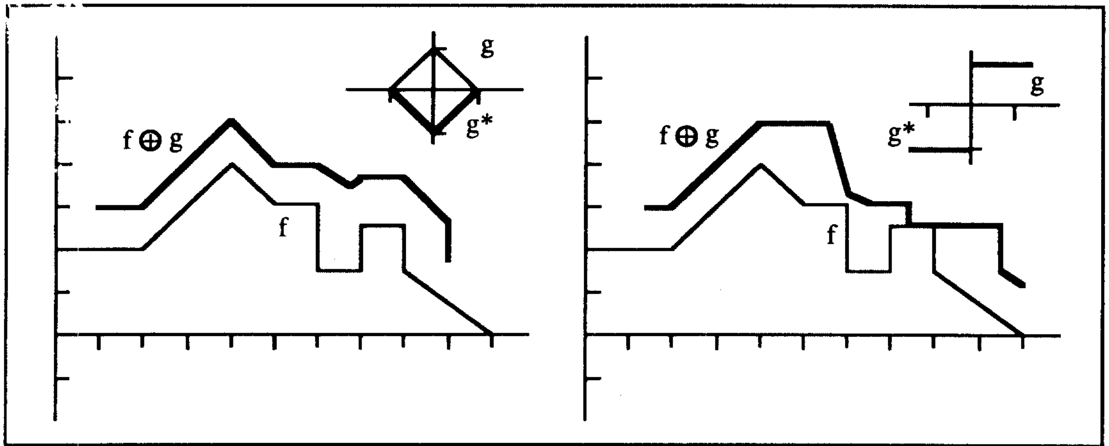

Fig. 2. An examples of gray-scale dilation.

Instead of finding the minimum "offset" at a point $\mathbf{x}$, as we did for gray-scale erosion, we can find the "maximum difference" between $f(\mathbf{z})$'s and $(g^*)_\mathbf{x}(\mathbf{z})$'s for all $\mathbf{z} \in D[(g^*)_\mathbf{x}]$. This notion leads to the formulation of dilation

$$(f \oplus g)(\mathbf{x}) = \max \{f(\mathbf{z}) - (g^*)_\mathbf{x}(\mathbf{z}) : \mathbf{z} \in D[(g^*)_\mathbf{x}]\}. \tag{4}$$

2.2. GENERALIZED-MEAN OPERATOR

The generalized mean operator is defined [Dujmovic, Dyckhoff] as

$$g(x_i; p, w_i) = \left[\sum_i w_i x_i^p\right]^{1/p} \tag{5}$$

where

$$\sum_i w_i = 1, \quad w_i > 0, \quad 0 \le x_i \le 1, \text{and } p \ne 0. \tag{6}$$

This operator has several attractive properties. For example, the mean value monotonically increases with respect to p when the w_i's are fixed [Dyckhoff]. Thus, by varying p from $-\infty$ to

$+\infty$, we can obtain all values between $\min\{x_i\}$ and $\max\{x_i\}$. This property was used to simulate linguistic concepts such as "at least" and "at most" by choosing appropriate values for the parameter p [Krishnapuram 88 & 92a, Lee]. The w_i's can be thought of as the relative importance factors for the different information criteria x_i's. This property was also used for redundancy detection for a hierarchical fuzzy information fusion system [Krishnapuram 92ab, Lee]. The generalized-mean value is equal to harmonic mean if p = -1, geometric mean if p = 0, and arithmetic mean if p = 1.

III. FUZZY EROSION AND DILATION

The assumptions in (6) can be kept as hard constraints. However, we use soft assumptions in our definitions in order to implement a neural network system later on [Won 95b]. The following theorem provides the motivation.

Theorem: Suppose we have a finite set $\{x_i;\ i = 1, 2, \cdots, N\}$. If $w_i > 0$ and $x_i \geq 0$ for all i, then

$$\lim_{p\to-\infty} g(x_i;\ p,\ w_i) = \min\{x_i\} \text{ and } \lim_{p\to+\infty} g(x_i;\ p,\ w_i) = \max\{x_i\}.$$

Proof: For any k,

$$g(x_i;p,w_i) = \left[\sum_i w_i x_i^p\right]^{1/p} = x_k\left[\sum_{i\neq k} w_i\left(\frac{x_i}{x_k}\right)^p + w_k\right]^{1/p} = x_k[f(p)]^{1/p}$$

where

$$f(p) = \sum_{i\neq k} w_i\left(\frac{x_i}{x_k}\right)^p + w_k$$

Let x_k be $\min\{x_i\}$. Then,

$$\lim_{p\to-\infty}\left(\frac{x_i}{x_k}\right)^p = \lim_{p\to+\infty}\left(\frac{x_k}{x_i}\right)^p = 0, \text{ because } 0 \leq \left(\frac{x_k}{x_i}\right) < 1.$$

Thus,

$$\lim_{p\to-\infty} f(p) = w_k.$$

Note that

$$\lim_{p\to-\infty}\frac{\ln f(p)}{p} = \lim_{p\to-\infty}\frac{\ln w_k}{p} = 0$$

Therefore,

$$\lim_{p\to-\infty}[f(p)]^{1/p} = \lim_{p\to-\infty}\exp\left\{\frac{\ln[f(p)]}{p}\right\} = \exp(0) = 1$$

and $\lim_{p\to-\infty} g(x_i;\ p,\ w_i) = x_k = \min\{x_i\}$.

Let x_k be $\max\{x_i\}$. Then,

$$\lim_{p\to+\infty}\left(\frac{x_i}{x_k}\right)^p = 0, \text{ because } 0 \leq \left(\frac{x_i}{x_k}\right) < 1.$$

In the same manner, we can show that $\lim_{p\to+\infty} g(x_i;\ p,\ w_i) = x_k = \max\{x_i\}$.

Q.E.D.

From this theorem, we can formalize the erosion and dilation with the generalized-mean operator. Assume, for a while, that $f(\mathbf{z})-h_{\mathbf{x}}(\mathbf{z}) \geq 0$ and $f(\mathbf{z})-(m^{*})_{\mathbf{x}}(\mathbf{z}) \geq 0$ for all $\mathbf{z}$. Let us use the notation $g(x_i;\, p = \pm\infty,\, w_i)$ to denote $\lim_{p \to \pm\infty} g(x_i;\, p,\, w_i)$. Then erosion and dilation can be represented using the generalized mean:

$$\begin{aligned}\text{Erosion} : (f \ominus g)(\mathbf{x}) &= \min\{f(\mathbf{z})-h_{\mathbf{x}}(\mathbf{z}) : \mathbf{z} \in D[h_{\mathbf{x}}]\} \\ &= g\{f(\mathbf{z})-h_{\mathbf{x}}(\mathbf{z});\, p = -\infty,\, w_i\} \qquad (7a)\end{aligned}$$

$$\begin{aligned}\text{Dilation} : (f \oplus g)(\mathbf{x}) &= \max\{f(\mathbf{z}) - (m^{*})_{\mathbf{x}}(\mathbf{z}) : \mathbf{z} \in D[(m^{*})_{\mathbf{x}}]\} \\ &= g\{f(\mathbf{z}) - (m^{*})_{\mathbf{x}}(\mathbf{z})\,;\, p = +\infty,\, w_i\}. \qquad (7b)\end{aligned}$$

To avoid the assumption that $f(\mathbf{z})-h_{\mathbf{x}}(\mathbf{z}) \geq 0$ and $f(\mathbf{z})-(m^{*})_{\mathbf{x}}(\mathbf{z}) \geq 0$ for all $\mathbf{z}$, we can use a one-to-one, increasing function $r:[-\infty, +\infty] \to [0\ +\infty]$. This modification yields modified definitions of erosion and dilation:

$$\text{Erosion} : (f \ominus_r g)(\mathbf{x}) = g\{r(f(\mathbf{z})-h_{\mathbf{x}}(\mathbf{z}));\, p = -\infty,\, w_i\} \qquad (8a)$$

$$\text{Dilation} : (f \oplus_r g)(\mathbf{x}) = g\{r(f(\mathbf{z})-(m^{*})_{\mathbf{x}}(\mathbf{z}))\,;\, p = +\infty,\, w_i\}. \qquad (8b)$$

Note that the weighting factor w_i's do not play a role in these definitions. These modified definitions have empirically shown that they behave similarly to ordinary ones with a sigmoid function for r [Gader 93a, Won 95a]. One can use other functions that satisfies the assumptions, such as a clipper function which was defined by

$$r(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1. \\ 1 & \text{if } x \geq 1 \end{cases} \qquad (9)$$

At this point, defining fuzzy erosion and dilation is straightforward. They are formalized as

$$\text{Fuzzy Erosion} : (f \ominus_f g)(\mathbf{x}) = g\{r(f(\mathbf{z})-h_{\mathbf{x}}(\mathbf{z}));\, p < 0,\, w_i\} \qquad (10a)$$

$$\text{Fuzzy Dilation} : (f \oplus_f g)(\mathbf{x}) = g\{r(f(\mathbf{z})-(m^{*})_{\mathbf{x}}(\mathbf{z}))\,;\, p > 0,\, w_i\}. \qquad (10b)$$

Note that the weighting factor w_i's are optional. If the factor is involved, the definition is a "weighted" fuzzy erosion and dilation.

IV. CONCLUSION

We have described a novel definition for fuzzy mathematical morphology. The generalized-mean operator played a key role in this work. Some hard constraints for the standard generalized-mean operator has been eliminated to simplify the definitions. Our definition is well suited for determining the parameters using neural network learning paradigm.

Therefore, the parameters for the fuzzy definition can be optimized using neural network learning paradigm.

The main goal of this paper was to introduce a new definition for fuzzy morphological erosion and dilation. We believe that further refinements of these definitions are necessary. Furthermore, a neural network implementation to design the structuring element and to determine the fuzzy membership parameter and the weighting factors is the most attractive study [Won 95b].

REFERENCES

J.C. Bezdek, *Pattern Recognition with Fuzzy Objective Function Algorithms*. New York: Plenum, 1981.

E.R. Dougherty, *An Introduction to Morphological Image Processing*, SPIE Optical Engineering Press, 1992.

A. Dujmovic, "Weighted Conjunctive and Disjunctive Means and Their Application in System Evaluation." *Publikacije Electrotechnickog Faculteta Beograd, Serija Matematika i Fizika*, Vol. 483, pp. 147 - 158, 1974.

H. Dyckhoff and W. Pedrycz, "Generalized Means as a Model of Compensation Connectives." *Fuzzy Sets and Systems*, Vol. 14, No. 2, pp. 143 - 154, 1984.

P.D. Gader, Y. Won and M.A. Khabou, "Image Algebra Networks for Pattern Classification," *SPIE Mathematical Morphology and Image Algebra*, pp. 157 - 168, July, 1993a.

C.R. Giardina and E.R. Dougherty, *Morphological Methods in Image and Signal Processing*. Englewood Cliffs, NJ, Pretice-Hall, 1988.

R.M. Haralick,S.R. Stenberg and X. Zhuang, "Grayscale Morphology", in *Proc. IEEE Conf. on Computer Vision and Pattern Recognition*, Miami FL, 1986.

H.J.A.M. Heijmans, "Theoretical Aspects of Gray-level Morphology.", *IEEE Trans. on Pattern Analysis and Machine Intelligence*, Vol. 13, No. 6, pp. 568 - 582, 1994.

R. Krishnapuram and J. Lee, "Propagation of Uncertainty in Neural Networks." in *Proc. of SPIE Conference on Robotics and Computer Vision*, Cambridge, MA, pp. 377 - 383, November, 1988.

R. Krishnapuram and J. Lee, "Fuzzy-Connective-Based Hierarchical Aggregation Networks for Decision Making." *Fuzzy Sets and Systems*, Vol. 46, No. 3, 1992a.

R. Krishnapuram and J. Lee, "Fuzzy-Set-Based Hierarchical Networks for Information Fusion in Computer Vision." *Neural Networks*, Vol. 5, pp. 335 - 350, 1992b.

J. Lee, "Fuzzy-Set-Connective-Based Information Fusion Techniques and Their Application to Computer Vision." Ph.D. Dissertation, University of Missouri-Columbia, 1990.

P. Maragos and R.W. Schafer, "Morphological filters - Part 1: Their Set-theoretic Analysis and Relations to Linear Shift-invariant filters.", *IEEE Trans. Acoust. Speech Signal Processing*, Vol. 35, pp. 1153 - 1169, 1987.

P. Maragos, "A Representation Theory for Morphological Image and Signal Processing.", *IEEE Trans. Pattern Analysis and Machine Intelligence*, Vol. 11, pp. 586 - 599, 1989.

G. Matheron, *Random Sets and Integral Geometry*, Wiley, New York, 1975.

Y. Nakagawa and Rosenfeld, "A Note on the Use of local Min and Max Operation in Digital Picture Processing." *IEEE Trans. on System, Man and Cybernetics*. Vol. 8, 1978.

J. Serra, *Image Analysis and Mathematical morphology, Vol. 1*, Academic Press, New York, 1982a.

J. Serra, *Image Analysis and Mathematical morphology, Vol. 2: Theoretical Advances,* Academic Press, New York, 1982b.

S.R. Sternberg, "Grayscale Morphology", *Computer Vision, Graphics, and Image Processing*, Vol. 35, pp. 333 - 355, 1986.

Y. Won, "Nonlinear Correlation Filter and Morphology Neural Networks for Image Pattern and Automatic Target Recognition." Ph.D. Dissertation, University of Missouri-Columbia, 1995a.

Y. Won and B. Lee, "A Novel Fuzzy Morphology, Part II: Neural Network Implementation." in *Korea Fuzzy Logic and Intelligent Systems*, see this proceedings, 1995b.

L.A. Zadeh, "Fuzzy Sets," *Informa. Contr.*, vol. 8, pp. 338 - 353, 1965.