AN EXTENSION THEORY OF FUZZY TOPOLOGICAL SPACES

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1. Introduction

In 1978, T.E. Gantner, R.C. Steinlage and R.H. Warren[2] construct one-point α-compactification of an L-fuzzy topological space, and in 1980, H.W. Martin[5] constructed a Stone-Čech ultra-fuzzy compactification for any fuzzy topological space whose induced topology is completely regular and he obtained some interesting properties of his compactification in a weakly induced fuzzy topological space. In this talk, we introduce a notion of an ultra-extension of a fuzzy topological space and construct such an ultra-extension via the existence of extension of a toplogical space. Moreover we show that the ultra-extension of a fuzzy topological space inherits some interesting properties of the given space. As examples, we obtain an ultra H-closed extension, an one-point H-closed extension and an ultra compactification. We note that these results are generalizations of those obtained by H.W. Martin[5,6,7]

2. Basic notions

For a topological space $(X, \mathcal{T})$, we have a fuzzy topological space $(X, \omega(\mathcal{T}))$, where $\omega(\mathcal{T})$ is the set of all lower semi-continuous maps from X into I. Let I_r be the unit interval I with the right open topology. For a fuzzy topological space (X, δ), we have a topological space $(X, \iota(\delta))$, where $\iota(\delta)$ is the initial topology on X with repect to the family $\{A : X \longrightarrow I_r\}_{A \in \delta}$.

Let **Top** denote the category of topological spaces and continuous maps and **FTop** the category of fuzzy topological spaces and fuzzy continuous maps. Then we have the following two functors

$$\omega : \mathbf{Top} \longrightarrow \mathbf{FTop}$$

where $\omega(X, \mathcal{T}) = (X, \omega(\mathcal{T}))$ and $\omega(f) = f$, and

$$\iota : \mathbf{FTop} \longrightarrow \mathbf{Top}$$

where $\iota(X, \delta) = (X, \iota(\delta))$ and $\iota(f) = f$. We note that $\iota \circ \omega = id$ and **Top** is a bi(co)reflective subcategory of **FTop** via ω.

Definition 2.1. Let (X, δ) be a fuzzy topological space. Then

(1) For $A \in I^X, A$ is *dense* in (X, δ) if $\bar{A} = X$.

(2) A is an *ultra-dense* in (X, δ) if $A^{-1}((\alpha, 1])$ is dense in $(X, \iota(\delta))$ for every $\alpha \in [0, 1)$.

Proposition 2.2. *A fuzzy set A in X is ultra-dense in (X, δ) if and only if A is dense in the induced space $(X, \omega(\iota(\delta)))$.*

Corollary 2.3. *If A is ultra-dense in (X, δ), then A is dense in (X, δ).*

Definition 2.4. A fuzzy topological space (X, δ) is *fuzzy ultra-Hausdorff* if $(X, \iota(\delta))$ is Hausdorff space.

Proposition 2.5. *If (X, δ) is a fuzzy topological T_2-space, then (X, δ) is ultra-Hausdorff.*

3. An ultra extension

Definition 3.1. Let (X, δ) and (X^*, δ^*) be fuzzy topological spaces and $e : X \longrightarrow X^*$ a map. Then the pair (X^*, e) is called an *extension* of (X, δ) if e is a dense embedding. In the following by an extension of (X, δ) we mean a fuzzy topological space (X^*, δ^*) of which (X, δ) is a dense subspace.

Theorem 3.2. *Let (X, δ) be a fuzzy topological space and $(X^*, \mathcal{T}^*)$ be a topological space which contains $(X, \iota(\delta))$ as a dense subspace. Let $\delta_{\mathcal{T}^*} = \{G \,|\, G : (X^*, \mathcal{T}^*) \longrightarrow$*

I l.s.c. map s.t.$G|X \in \delta\}$. Then $(X^*, \delta_{\mathcal{T}^*})$ is a fuzzy topological space and (X, δ) is an ultra-dense fuzzy topological subspace of $(X^*, \delta_{\mathcal{T}^*})$.

Definition 3.3. Let (X, δ) and (X^*, δ^*) be fuzzy topological spaces. Then (X^*, δ^*) is called an *ultra-extension* of (X, δ) if there exists an extension $(X^*, \mathcal{T}^*)$ of $(X, \iota(\delta))$ such that $\delta_{\mathcal{T}^*} = \delta^*$.

Remark. Let (X, δ) be a fuzzy topological space and let δ_c be the set of all characteristic maps in δ. Then δ_c is a fuzzy topology on X. Define $\delta_c^* = \{A \subseteq X | 1_A \in \delta_c\}$. Then (X, δ_c^*) is a topological space.

Definition 3.4. A fuzzy topological space (X, δ) is said to be an *induced fuzzy topological space* (or an *induced space*) if δ is the collection of all lower semi-continuous maps from (X, δ_c^*) into I. i.e. $\delta = \omega(\delta_c^*)$.

Definition 3.5. A fuzzy topological space (X, δ) is said to be a *weakly induced space* if, whenever $G \in \delta$, then $G : (X, \delta_c^*) \longrightarrow I$ is a l.s.c. map. i.e. $\delta \subseteq \omega(\delta_c^*)$.

Theorem 3.6. *Every fuzzy compact Hausdorff space is a weakly induced space.*

Theorem 3.7. *A fuzzy topological space (X, δ) is an induced space iff (X, δ) is a weakly induced space in which every constant maps from X into I belongs to δ.*

Theorem 3.8. *Let (X, δ) be a fuzzy topological space and $(X^*, \delta_{\mathcal{T}^*})$ an ultra fuzzy extension of (X, δ). Then the following holds.*

(1) *The constant maps from X into I belong to δ iff the constant maps from X^* into I belong to $\delta_{\mathcal{T}^*}$.*

(2) *The members of δ are characteristic maps iff the members of $\delta_{\mathcal{T}^*}$ are characteristic maps.*

(3) *The space (X, δ) is weakly induced iff $(X^*, \delta_{\mathcal{T}^*})$ is weakly induced.*

(4) *The space (X, δ) is induced iff $(X^*, \delta_{\mathcal{T}^*})$ is an induced space.*

References

[1] C.L. Chang, *Fuzzy topological spaces*, J. of Math. Analysis and Applications, 24(1966) 182-190.

[2] T.E. Gantner and R.C. Steinlager and R.H. Warren, *Compactness in Fuzzy Topological Spaces*, J. of Math. Analysis and Applications 62, (1978) 547-562.

[3] R. Lowen, *Fuzzy Topological Spaces and Fuzzy Compactness*, J. of Math. Analysis and Applications 56, (1976) 621-633.

[4] R. Lowen, *Initial and Final Fuzzy Topologies and the Fuzzy Tychonoff Theorem*, J. of Math. Analysis and Applications 58, (1977) 11-21.

[5] H.W. Martin, *A Stone-Čech Ultrafuzzy Compactification*, J. of Math. Analysis and Applications 73, (1980) 453-456.

[6] H.W. Martin, *Weakly Induced Fuzzy Topological Spaces*, J. of Math. Analysis and Applications 78, (1980) 634-639.

[7] H.W. Martin, *A Characterization of Fuzzy Compactifications*, J. of Math. Analysis and Applications 133, (1988) 404-410.