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# 분기 모우드를 활용한 얇은 빔의 혼돈 역학에 관한 연구

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(On the Chaotic Vibrations of Thin Beams by a Bifurcation Mode)

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## 1. INTRODUCTION

The static and dynamic stability of a thin beam have been widely investigated by many researchers, both theoretically and experimentally. The lateral instability under static loads is well known as a "lateral buckling" of beams[1] and has been studied in elastic stability problems. Dynamic instability of simple structures under periodic loads has been extensively treated by many investigators in a theoretical way[2].

Experimental observations[3, 4] provide many interesting nonlinear dynamic features in a thin beam, but there are only a few analytical methods reported to explain them. Pak[5] recently utilized harmonic balance method(HBM)[6] for the analysis of the lateral stability of a thin beam under periodic bending loads. The eigenfunctions of bifurcation modes, which play a role of the seeds of newly-borne motion, are utilized for HBM and that application produces satisfactory results in the analysis of the dynamic lateral instability of a thin elastica. Further, the natural forcing function was conceptually defined as the variation of the eigenfunctions with respect to system parameters. The introduction of the so-defined natural forcing function provided a very convenient way of calculating the responses of undamped and damped forced vibrations.

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The main objective of this work is to study the chaotic dynamics of a thin beam under periodic bending loads by one of bifurcation modes(called "nonlocal mode : NLM[7]") which causes lateral instability.

The calculation of fractal dimension confirms us that a thin elastica, which has infinite degrees of freedom since it is a continuous system, can be modeled into a low-dimension one to show the chaotic dynamics of the system[3]. Hence the elastica is modeled into a two degree-of-freedom system composed of the first bending and the first torsional mode.

Nonlinear dynamic characteristics of the elastica are generally investigated by Pak et al[8]. The easily-known periodic motions of the elastica are torsion and bending and their stability is examined by the perturbation techniques. From the perturbation analysis, several periodic bifurcation modes are found : nonlinear normal mode, elliptic orbit mode, and nonlocal mode.

Since the lateral instabilization and hence the chaotic vibration are of interest, the nonlocal mode which is generated due to stability change in torsional mode is computed with the aid of HBM. Firstly evaluation of the stability of periodic motions is generally overviewed to apply the Synge's concept[8], and then the free vibration features of the bifurcation mode are examined. By the natural forcing function, the responses are natural when damping and forcing amplitude are weak. With the increment of them, the expected responses are shown for narrower ranges of frequency.

Keeping the fact in mind that the computation is quite reasonable for somewhat large damping and forcing, we will testify the route-to-chaos of the system. The route seems to be composed of quasi-periodic(Hopf bifurcation) and mainly of period doubling process which can be easily conjectured from the first generation of the bifurcation mode(pitchfork bifurcation). With further process of bifurcation, there may appear the dynamic two-well behavior[3] similarly as in a buckled-beam system[9]. As verifications, Poincaré maps, FFT's, and Lyapunov exponents are employed.

## 2. SYSTEM

Consider a uniform and straight elastic rod which can be bent and twisted(Fig. 1). For small motions of the rod, the linear theory is applicable ; bending and torsion can be decoupled so that solutions are obtained by the principle of superposition. It is however not the case for large motions. When the motion becomes large, bending and torsion are coupled by geometrical nonlinearity (due to curvature and inertia effects). Hence the exact and explicitly-written equations of motion are not available.

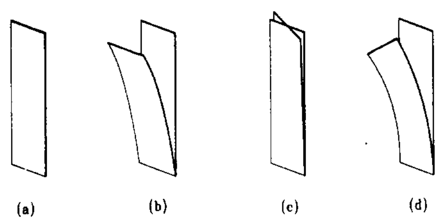


Fig. 1 (a) A uniform straight elastic rod (b) Bending motion (c) Torsion motion (d) Coupled bending- torsional motion

According to Cusumano, it was assumed that a thin rod can be bent only in the flexible direction. With a proper simplification and neglect of nonlinear curvature effect, the two coupled bending-torsional nonlinear partial differential equations were derived. Fractal dimension analysis made from his experimental data showed that the chaotic dynamics of the elastica were low-dimensional. Hence it was good enough to model the system with two degrees of freedom. By Galerkin procedure and with a simple mechanical analogue, the system can finally be modeled as in Fig. 2.

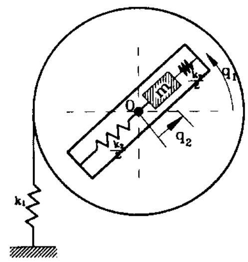


Fig. 2 A simple mechanical analogue to elastica

The kinetic energy T and the potential energies are as what follows ($\tau$ : time) :

$$T=\frac{m}{2}(q_2^2 q_1'^2+q_2'^2)+\frac{J}{2}q_1'^2,\qquad V=\frac{1}{2}(k_1 q_1^2+k_2 q_2^2),\quad '=\frac{d}{d\tau} \tag{1}$$

Through the non-dimensionalization by using the concept of the characteristic length $\delta$ defined as $\varepsilon=\sqrt{(m\delta^2/J)}$ ($\varepsilon$ : a nondimensional parameter), the non-dimensional T and V can be written as

$$T=\frac{1}{2}(1+\varepsilon y^2)\,\dot{x}^2+\frac{1}{2}\,\dot{y}^2,\qquad V=\frac{1}{2}(p^2x^2+y^2) \tag{2}$$

where $x=\sqrt{\{J/(m\delta^2)\}}q_1=q_1/\varepsilon$ is the torsional displacement, $y=q_2/\delta$ the bending displacement, and $p = (k_1/J)/(k_2/m)$ the ratio of linearized natural frequencies.

## 3. COMPUTATION OF NLM AS A BIFURCATION MODE

### 3.1 Free Vibrations and a Bifurcation Mode

Write the equations of motion from Eq.(2) as

$$\begin{aligned}(1+\varepsilon y^2)\ddot{x}+2\varepsilon y\dot{y}\dot{x}+p^2x &= 0\\ \ddot{y}-\varepsilon\,\dot{x}^2y+y &= 0.\end{aligned} \tag{3}$$

Since the lateral instability is closely related with the stability change of the x-mode, consider its stability. Write the x-mode as

$$x=A\sin pt+\eta,\ \ y=0+\xi \tag{4}$$

Then the variational equations for the x-mode are

$$\begin{aligned}&\ddot{\eta}+p^2\eta=0\\ &\ddot{\xi}+(1-\varepsilon A^2p^2\cos^2 pt)\xi=0.\end{aligned} \tag{5}$$

The eigenvalues of Eq.(5 a) are $\lambda_{1,2} = \pm\, ip$ ($i^2 = -1$)

and the remaining two eigenvalues determine the stability of the periodic motion(in this case, x-mode). We know that the Hamiltonian systems have two characteristic exponents which are identically zero due to the symplectic structure[10] and hence it will be very reasonable to use the Synge's stability concept in the kinematico-statical sense[8]. The resultant equation for the magnitude of the disturbance vector becomes the standard Mathieu equation[6]

$$\ddot{\beta}+(\delta+2e\cos 2\tau)\beta=0,\ \tau=pt. \tag{6}$$

where $\delta=1/p^2-\varepsilon A^2/2$, and $e=-\varepsilon A^2/4$. The bifurcation mode occurs when the arrowed line $\delta = 1/p^2 + 2e$ (x-mode) crosses the transition curve $\delta = -e^2/2 + O(e^2)$, as shown below. This corresponds to $h = T + V = 10.0$, when the parameters are $p = 20.0$ and $\varepsilon = 0.1$(Fig. 3). It is stable for small amplitude A and gets unstable when $A^2 \geq 2/(\varepsilon p^2)$(when $p \geq 1$). The eigenfunction [6] corresponding to this transition curve is written as

$$\vec{\beta}(t) = C(1+e\cos 2pt/2)+O(e^2) \tag{7}$$

where $\vec{\beta}(t)$ is in the y-direction and C is an arbitrary constant. This eigenfunction will be the seed for the bifurcation mode, called NLM, and will be implemented in forced vibration of a bifurcation mode.

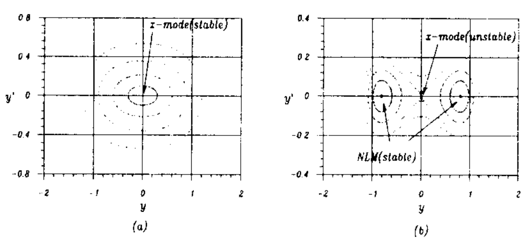


Fig. 3 Bifurcation of the x-mode into NLM (a)stable x-mode(h = 9.0) (b) unstable x-mode and two stable nonlocal mode(h = 11.0)

Let $\vec{x}(t)$ be the solution of perturbed trajectory at the stability change. Then it can be written as

$$\vec{x}(t)=\vec{x}^*(t)\pm\vec{\beta}^*(t), \tag{8}$$

where $\vec{x}^*(t)$ is the unperturbed periodic solution and $\vec{\beta}^*(t)$ the eigenfunction of the bifurcation-yielding transition curve.

With the further increase in total energy, the above equation can be assumed to be the generating function to describe the bifurcation mode. Decomposing this function into each generalized coordinate function of the form

$$\vec{x}_i(\omega t)=\vec{x}_i^*(\omega t)\pm\vec{\beta}_i^*(\omega t),\ i=1,2. \tag{9}$$

When it is substituted into the equations of motion, the harmonic balance method(HBM) can be applicable.

NLM can be expressed as infinite series of harmonics

$$x(t)=\sum_{n=1}^{\infty}A_{2n-1}\sin(2n-1)\omega t,$$
$$y(t)=B_0+\sum_{n=1}^{\infty}B_{2n}\cos 2n\omega t \tag{10}$$

Use the first approximation of NLM, then it will the same form as above :

$$x(t)=A\sin\omega t,\ y(t)=B_0+B\cos 2\omega t \tag{11}$$

Note that by Eq. (8), we have

$$x_1^*(t)=A\sin\omega t,\ x_2^*(t)=0,$$
$$\beta_1^*(t)=0,\ \beta_2^*(t)=B_0+B\cos 2\omega t. \tag{12}$$

Substituting into Eq. (3) and balancing harmonics, we have the following three equations.

$$A\left\{p^2-\omega^2-\varepsilon\omega^2(B_0^2+B_0B+\frac{1}{2}B^2)\right\} = 0\ (A\neq 0),$$
$$B_0-\frac{\varepsilon}{2}\omega^2A^2(B_0+\frac{B}{2}) = 0, \tag{13}$$
$$(1-4\omega^2)B-\frac{\varepsilon}{2}\omega^2A^2(B_0+B) = 0.$$

Backbone curves from Eq. (13) are shown in Fig. 4.

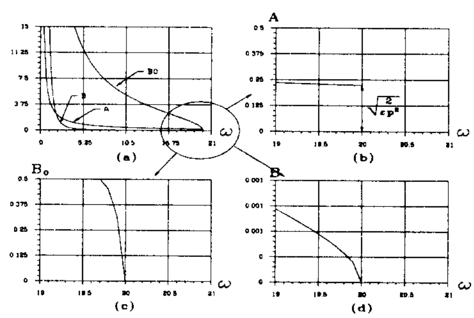


Fig. 4 (a) Backbone curves of NLM (b), (c), (d) The region near p is magnified

### 3.2 Applications to Forced Vibrations : Natural Forcing Function

By utilizing the procedure of computing bifurcation modes, forced responses can be calculated for weakly damped forced systems. Let $\alpha_i$, i = 1, 2, ..., N, be mutually independent systems parameters including h. Assume that the homoclinic loops, which occur with the stability change of periodic motion, are persisted under small variations in $\alpha_i$'s. Then define the natural forcing function[11] $\overrightarrow{f(t)}$ in the form

$$\overrightarrow{f(t)} = \sum c_i \frac{\partial \overrightarrow{\beta^*(t)}}{\partial \alpha_i} \Delta \alpha_i \tag{14}$$

where $\Delta\alpha_i$ is a small variation in $\alpha_i$, and $c_i$ arbitrary constant. When $\overrightarrow{f(t)}$ is added to the equations of motion, a varied bifurcation mode will be obtained, which can be regarded as a forced response. Since the eigenfunction $\beta^*(t)$ is expressed in a series of infinite harmonics, the natural forcing function is not practical. However, the first few terms are dominant, and we can choose the natural forcing function as practical as possible in the N-dimensional vector space defined by the above equation, implying a wide variety of choice.

For weakly damped systems, terms having high harmonics may be dissipated so that a periodic response is expected under simple harmonic excitations. When a practical natural force is applied to damped systems, it is possible to obtain stable periodic solutions other than those associated with bifurcation modes. In this case, each solution has its own domain of attraction.

#### 3.2.1 Undamped Forced Vibrations

According to the concept of natural forcing function, for the NLM response as a bifurcation mode, the following function should be applied to the system.

Remind that the eigenfunction of the transition curve which generates the NLM is (Also note that is in the y-direction.)

$$\vec{\beta}(t) = C(1 + \frac{e}{2}\cos 2pt) + O(e^2). \tag{6}$$

Since e = $-\varepsilon A^2/4$, the eigenfunction becomes(when p is replaced with $\Omega$)

$$\vec{\beta}(t) = C(1 - \frac{\varepsilon A^2}{8}\cos 2\Omega t) + O(\varepsilon^2). \tag{15}$$

The derived forcing function becomes

$$\therefore\ f(t) = C_1 C(-\frac{A^2}{8})\cos 2\Omega t \equiv F_0 \cos 2\Omega t \tag{16}$$

Since $\vec{\beta}(t)$ is in the y-direction, also so is the forcing function. Hence the equations of motion in undamped forced vibration becomes

$$\begin{aligned}(1+\varepsilon y^2)\ddot{x} + 2\varepsilon y\dot{y}\dot{x} + p^2 x &= 0,\\ \ddot{y} - \varepsilon \dot{x}^2 y + y &= F_0 \cos 2\Omega t.\end{aligned} \tag{17}$$

In order to apply the HBM, assume that solutions are

$$x(t) = A\sin\Omega t,\ \ y(t) = B_0 + B\cos 2\Omega t. \tag{18}$$

The balancing result can be written as :

$$\begin{aligned}A\left\{p^2 - \Omega^2 - \varepsilon\Omega^2(B_0^2 + B_0 B + \frac{1}{2}B^2)\right\} &= 0,\\ B_0 - \frac{\varepsilon}{2}\Omega^2 A^2(B_0 + \frac{B}{2}) &= 0,\\ (1-4\Omega^2)B - \frac{\varepsilon}{2}\Omega^2 A^2(B_0 + B) &= F_0.\end{aligned} \tag{19}$$

There are two case with respect to the value of A.

(i) A = 0 : Single mode response

In this case, $B_0$ = 0 from the second of Eq. (18).

Hence the frequency response equation becomes

$$B = F_0/(1-4\Omega^2) \tag{20}$$

The corresponding solutions are

$$x(t) = 0,\ \ y(t) = \frac{F_0}{1-4\Omega^2}\cos 2\Omega t. \tag{21}$$

(ii) A≠0 : Coupled mode response

The responses are of the form Eq. (18) where A, $B_0$, and B by Eq. (19).

The stability of the single mode response(case(i)) can be determined similarly as in the free vibration of the y-mode, if it is allowed to replace p with p/(4Ω). When 2π/(p/4Ω) is near 4π/N, or when Ω is near p/(2N), N = 1, 3, 5, ..... , the single mode response becomes unstable(Fig. 5 : p is replaced by p/4Ω and E by $\varepsilon F_0^2/(2(1-4\Omega^2)^2)$ ).

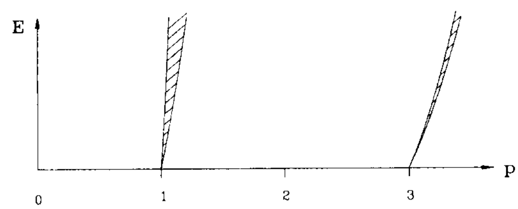


Fig. 5 Stability chart of y-mode

#### 3.2.2 Damped Forced Vibrations

The linear fluid dampings $2p\zeta_1\dot{x}$ and $2\zeta_2\dot{y}$ are added to the equations of motion in undamped forced vibrations.

Actually, it can be regarded as the structural damping of the beam.

The equations of motion then become

$$(1+\varepsilon y^2)\ddot{x} + 2\varepsilon y\dot{y}\dot{x} + 2p\zeta_1\dot{x} + p^2x=0,$$
$$\ddot{y} - \varepsilon\dot{x}^2y + 2\zeta_2\dot{y} + y=F_0\cos 2\Omega t \qquad (22)$$

Assume that solutions are

$$x(t)=A\sin(\Omega t-\phi_1),$$
$$y(t)=B_0+B\cos(2\Omega t-\phi_2). \qquad (23)$$

Substitution into the equations of motion yields the five equations which are obtained by balancing the harmonics and the analysis is quite similar as in undamped forced vibrations.

## 4. ROUTES-TO-CHAOS

### 4. 1 Identification Of Chaos

Since there are no generally accepted definition of chaos, it is appropriate to use the following working definition for practical purpose(12).

If the following ingredients are present ;

1. The underlying dynamics is deterministic ;

2. No external noise has been introduced ;

3. The seemingly erratic behaviour of individual trajectories depends sensitively on small changes of initial conditions ;

4. In contrast to a single trajectory, some global characteristics, obtained by averaging over many trajectories or over long time, e.g., a positive Lyapunov exponent, does not depend on initial conditions ;

5. And, when a parameter is tuned, the erratic state is reached via a sequence of events, usually including the appearance of one or more subharmonics ;

then one may well be dealing with chaos.

### 4.2 Chaotic Vibrations of Thin Beams : Route-to-Chaos

The selected parameters for the route-to-chaos are

$$\varepsilon=0.1,\ p=20,\ \zeta_1=\zeta_2=0.01,\ \Omega=4.25.$$

Before we go ahead, the choice of $\Omega$ is considered. Previously we knew that stability of the single mode response can be deduced from that of y-mode.

Since we choose p = 20 and $\Omega$ = 4.25, we get

$$p/2\Omega = 20/2\times4.25 \fallingdotseq 2.35,$$

or $p/(2\Omega)$ is neither close to any odd integer N. Therefore the stability changes of the single mode cannot make any effect on the bifurcation behaviors for the present choice of parameters.

The numerical integration technique used here is the simple fourth order Runge-Kutta method with step size of $\tau/10{,}000 \sim \tau/20{,}000$ where $\tau$ is a forcing period(when $2\Omega$ = 8.5, $\tau \simeq 0.739$. Thus the used step size is of order $10^{-4} \sim 10^{-5}$). The calculations are made up to $5\times10^3$(when $F_0<1000$), $1\times10^4$(when $1000<F_0<5000$), and $1.5\times10^4$(when $5000<F_0<20000$) for the steady state response. The initial conditions are fixed at the values where the nonlocal mode response can occur for $F_0 \simeq 7.0$, and the choice is made by considering the energy balance of the system. Poincaré sections are selected as what follow. If the equations of motion may be written as an autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ where $\mathbf{x} = \{x, x', y, y', \theta\}^T$, time t is replaced by $\theta$ as a new coordinate, and $\mathbf{x}\in\mathbf{R}^4\times S^1$($S^1=\mathbf{R}/\tau$ is the circle of length $\tau=2\pi/2\Omega$), then a cross section

$$\Sigma = \{(x, x', y, y', \theta)|\theta=0\}$$

can be picked. Define the Poincaré map

$$P : \Sigma \rightarrow \Sigma.$$

#### 4.2.1 General Descriptions

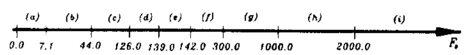


Fig. 6 Schematic representation of the ranges of the forcing amplitude $F_0$

● Range (b)

Nonlocal mode response occurs for the first time at $F_0$ = 7.1 where y(t) is of period 1 and x(t) of period 2, as expected. The nonlocal mode response becomes quasi-periodic at about $F_0$ = 20.0. As a specific characteristic of Hopf bifurcation, a fixed point forms a limit cycle. On the Poincaré maps, the fixed point appears as one point, and the limit cycle as a set of points with a circle-like formation. With more increment of forcing amplitude up to $F_0$ = 44.0, the quasi-periodic behavior grows larger, or the loops on the Poincaré maps become large.

● Range (d)

The nonlocal mode response abruptly occurs at $F_0$ =

126 and the motions are periodic(torsion : period 88, bending : 44) or quasi-periodic(a limit cycle). This motion grow quasi-periodic with the increment of forcing amplitude. The quasi-periodic motion is formed with the fixed point at $F_0$ = 126 as a base.

- Range (f)

  There dominantly exists the nonlocal mode response where x(t) is of period 8, and y(t) of period 4. Though it is suspected that there might exist the nonlocal mode response where x(t) is of period 4 and y(t) of period 2, it cannot be found.

- Range (g)

  There happens a period-doubling bifurcation. The nonlocal mode response has x(t) of period 16 and y(t) of period 8.

- Range (h)~

  With the further period-doubling process, chaotic vibrations occur.

#### 4.2.2 Results of Numerical Simulations

In this simulations, several forcing amplitudes are selected according to the region showing different dynamic behaviors : (a)$F_0$ = 8(region a), (b)$F_0$ = 44 (region b), (c)$F_0$ = 142(region f), (d)$F_0$ = 100000.

In Fig. 7, there are shown Poincaré maps in the x-x' and y-y' planes, time responses of x(t) and y(t), Lyapunov exponents, and FFT's of x(t) and y(t) from the top.

The first two rows well describe the route-to-chaos on the Poincaré maps : period doubling bifurcations containing Hopf bifurcations. Periodic motions(a) where x(t) is of 2τ, y(t) is of τ become limit cycles(b) of which a point form a set of points arranged in a circle. With the increment of forcing amplitude period-doubled motions are found : in (c), period-8τ x(t) and period-4τ y(t) are properly shown. Finally in (d), chaotic vibration is attained through further period-doublings, especially on the y-y' Poincaré map, dynamic two-well behavior is found which is similar to that in a buckled-beam[9].

The process can also be verified by FFTS of the responses[13, 14]. In the FFT's of y(t), the rigid portion is ignored which is shown as a one-sided response(the sixth and seventh ones). Here f corresponds to forcing frequency which is $f = 2\Omega/2\pi = 4.25/\pi = 1.353$. There are only on peak at 0.676(=1.353/2) for x(t) and at 1.353 for y(t) which implies period 2τ and τ motions in (a). In (b), there are shown sidebands on both sides of peak values where the spacing corresponds to the natural frequency of bending mode(f1)(note that the natural frequency of bending mode is $1/2\pi$=0.159 and this is the first peak in FFT of y(t)). (c) corresponds to the period 8τ and 4τ motion of x(t) and y(t), respectively. The peaks are shown in a sequence of f/8, 3f/8, 5f/8, ..., in FFT of x(t). In FFT of y(t), the peaks are shown in a series of f/4, 2f/4, 3f/4, f, ..., which implies period 4τ motion. At choas, broad spectrum of FFT's are obtained which means that all frequencies are excited.

Those all processes are also presented in time responses(the third and fourth rows). As a verification of the occurrence of chaos, the well-known Lyapunov exponents are calculated for $F_0$=10.0 and for $F_0=10^4$. The calculation method employs the repeated use of GSR(Gram-Schmidt Renormalization) procedure to prevent the divergence of the spectrum[15]. When $F_0$=10.0, which corresponds to the period 2τ and τ motion of x(t) and y(t), respectively, the calculated Lyapunov exponent $\lambda_i$'s are $(0, 0, -1.25\times10^{2}, -1.45\times10^{2})$ bits/sec. $\lambda = 0$ means that principal axes are on the average preserves in their magnitudes and negative exponents imply that the corresponding principal axis are on the average contraction. Globally, since $\Sigma\lambda_i < 0$, the volume in the phase space is continuously contraction, and it is the case because the system is dissipative. Physically there remain only periodic motions corresponding to $\lambda_i$ = 0(four dimensional phase space finally becomes two torus composed only of x(t) and y(t). When $F_0=10^4$ where the system is chaotic, the calculated Lyapunov exponents are $(6.75\times10^{2}, 2.46\times10^{2}, -2.67\times10^{2}, -6.78\times10^{2})$ bits/sec. At least one positive Lyapunov exponent implies chaos.

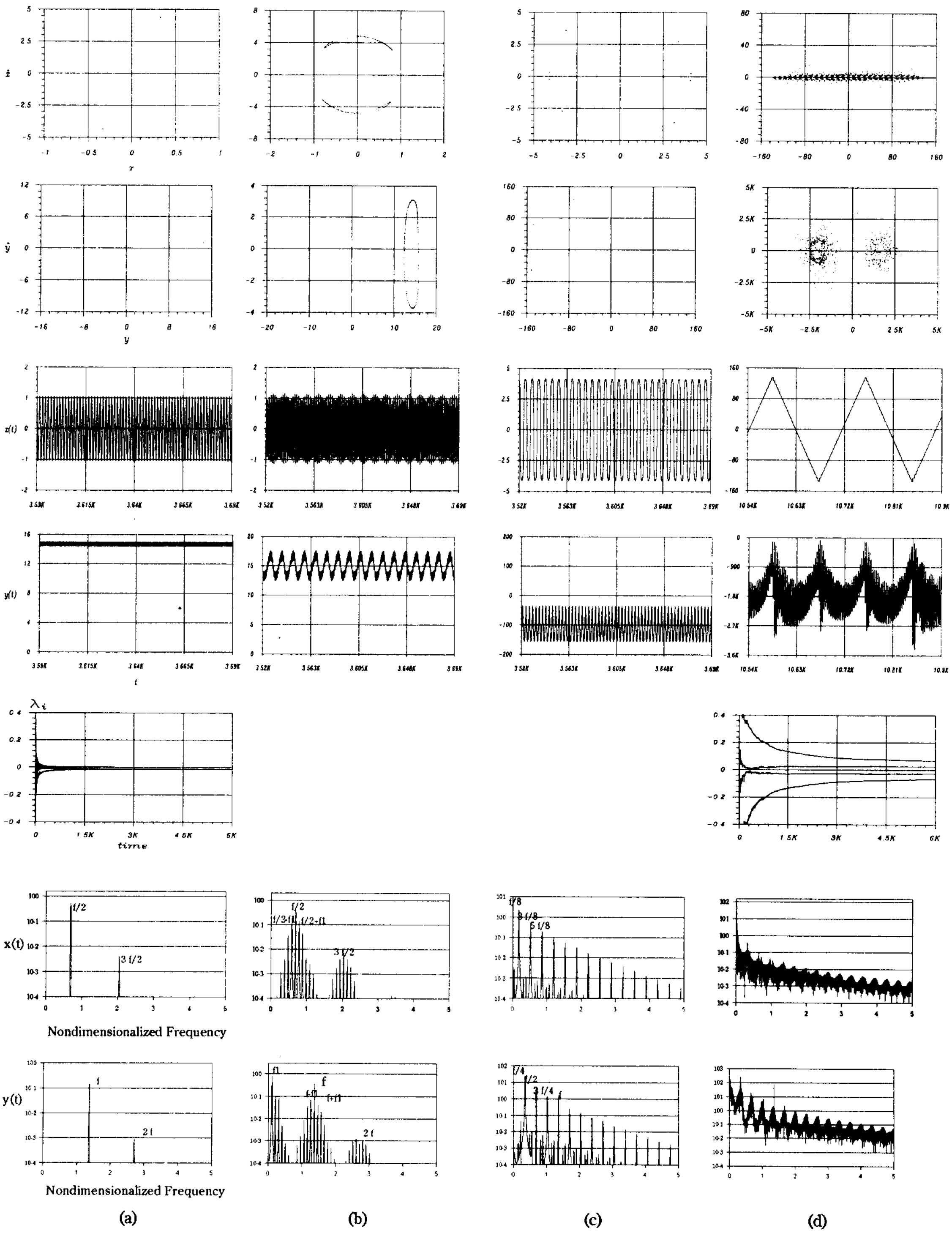


Fig. 7 Numerical results (a)$F_0$ = 8.0 (b)$F_0$ = 44.0 (c)$F_0$ = 142.0 (d)$F_0$ = $10^4$

Hence when $F_0=10^4$, chaos occurs in the system. The magnitude of Lyapunov exponent is much related to the quantification of an attractor's dynamics on information loss, or Lyapunov exponents measure the rate at which system processes create or destroy information.

## 5. CONCLUSIONS

The results are summarized as what follows.

1) The modeling of thin beams, which is a continuous system, into a two DOF system yields satisfactory results for the chaotic vibrations.
2) The concept of "natural forcing function" derived from the eigenfunction of the bifurcation mode is very useful for the natural responses of the system.
3) Among the perturbation techniques, HBM is a good estimate for the response when the geometry of motion is known.
4) It is known that there exist periodic solutions of coupled mode response for somewhat large damping and forcing amplitude, as well as weak damping and forcing.
5) The route-to-chaos related with lateral instability in thin beams is composed of period-doubling and quasiperiodic process and finally follows discontinuous period-doubling process.
6) The chaotic vibrations are verified by using Poincaré maps, FFT's, time responses, trajectories in the configration space, and the very powerful technique Lyapunov characteristic exponents.

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