한국전산구조공학회
1994년도 봄학술발표회 논문집

A PATH-SWITCHING STRATEGY BY COMBINING THE USE OF GENERALIZED INVERSE AND LINE SEARCH

K.K.CHOONG*, Y.HANGAI** and T.J.KWUN***

ABSTRACT

A path-switching strategy by combining the use of generalized inverse and line search is proposed. A reliable predictor for the tangent vector to bifurcation path is first computed by using the generalized inverse approach. A line search in the direction of maximum gradient of total potential at the point of intersection between the above predictor and a constant loading plane introduced in the vicinity of the detected bifurcation point is then carried out for the purpose of obtaining an improved approximation for a point on bifurcation path. With this approximation obtained, an actual point on bifurcation path is then computed through iteration on the constant loading plane.

1 INTRODUCTION

In order to check whether an engineering structure is prone to bifurcation buckling or not, we have to i) first determine the location of bifurcation point with respect to limit point on the equilibrium path and ii) next determine the shape of bifurcation path with respect to the primary path. The above two steps constitute what is commonly termed as bifurcation analysis. Bifurcation analysis is necessary when one wishes to investigate the imperfection sensitivity of the load carrying capacity of a given structure. The second step in bifurcation analysis is commonly called path-switching or branch-switching which is a computational process of obtaining a point lying on the bifurcation path. Assuming that the location of bifurcation point has been detected, then the associated bifurcation path could be traced as long as a vector representing the direction where the path is extending is known. The evaluation of this so-called switching-direction predictor is one of the core issues in any attempt for path-switching. Another major difficulty is encountered in the task of deciding the magnitude of this switching-direction predictor in such a way that a reasonably good approximation for an actual point lying on the bifurcation path could be obtained(Fig.1).

Fujii and Choong[refs 1,2] have proposed a line search process for locating an initial approximation for a point on bifurcation path. Provided that the search direction is appropriately chosen, this line search approach will yield a good approximation for a point lying on the bifurcation path. On the other hand, by using generalized inverse, Hangai et al[refs 3-5] have shown that a reliable switching-direction predictor could be obtained in a computationally neat way.

In this paper, the possibility of combining both the generalized inverse and line search approach to form an alternative path-switching strategy has been studied. After a brief review of

* Graduate Student, University of Tokyo, Tokyo, Japan
** Professor, Institute of Industrial Science, University of Tokyo, Tokyo, Japan
*** Professor, Sung Kyun Kwan University, Seoul, Korea

the available path-switching methods, the proposed strategy will then be described. Results of path-switching carried out on a numerical example is next presented. Further areas to be improved will also be pointed out. In this study, only distinct bifurcation problem of geometrically nonlinear elastic structures subjected to single parameter conservative proportional loading will be treated.

2 PATH-SWITCHING IN BIFURCATION ANALYSIS

Bifurcation path could be obtained by path-tracing once a point lying on it is obtained. An initial approximation for a point lying on the bifurcation path $\mathbf{X}_{II}(=\{\mathbf{v}_{II}, \lambda_{II}\}^T)$ could be obtained by adding a suitably chosen vector representing the switching-direction predictor to the detected bifurcation point $\mathbf{X}_B(=\{\mathbf{v}_B, \lambda_B\}^T)$ as follows

$$
\begin{aligned}
&X_{II}=X_B+\Delta\eta\dot{x}_{II}\\
&\dot{x}_{II}=\frac{\alpha_1\dot{x}_I+\alpha_2 b}{\|\alpha_1\dot{x}_I+\alpha_2 b\|}\quad,\quad \dot{x}_{II}=\{\dot{v}_{II},\dot{\lambda}_{II}\}^T
\end{aligned}
\tag{1}
$$

where $\dot{\mathbf{x}}_{II}$: linear approximation for switching-direction predictor, $\dot{\mathbf{x}}_I$: tangent vector to the primary path, $\mathbf{b}$: vector representing the bifurcation mode, α_1, α_2 : magnitude of $\dot{\mathbf{x}}_I$ and $\mathbf{b}$ respectively, $\Delta\eta$: magnitude of $\dot{\mathbf{x}}_{II}$, $\mathbf{v}, \lambda$: generalized displacement vector and loading parameter respectively, $\dot{(\;)}$: d( )/dη, η : path parameter and superscript T : notation for transpose of vector or matrix. Eigenvector corresponding to the smallest eigenvalue for tangent stiffness matrix at bifurcation point is used in most of the methods proposed in order to compute $\dot{\mathbf{x}}_{II}$[refs 6-9]. In [refs 1,2], alternative strategies whereby no eigenproblem is involved have been proposed. Methods where second order perturbation equations for equilibrium equations are used in the determination of $\dot{\mathbf{x}}_{II}$[refs 3-5,12,13] are comparatively fewer than those that make use of only linear incremental stiffness equations together with additional constraints[refs 1,2,10,14]. As for the determination of $\Delta\eta$, trial-and-error approach is the most commonly used method. Methods whereby the $\Delta\eta$ could be computed directly have also been proposed[refs 1,2,10]. For attaining a point on the bifurcation path from the initial approximation, iteration with various special constraints introduced for the purpose of ensuring convergence onto path other than the already known primary path is often used[refs 6,8,10,14,17]. A review on various methods for path-switching could be found in [ref.19].

2.1 Path-switching strategy by using generalized inverse[refs 3-5]

The set of equilibrium equations, first and second perturbation of equilibrium equations could be expressed respectively as follows

$$r(v,\lambda)=f(v,\lambda)-\lambda p_o=0 \tag{2}$$

$$r_{i,j}\dot{v}_j+r_{i,\lambda}\dot{\lambda}=0 \tag{3}$$

$$r_{i,j}\ddot{v}_j+r_{i,\lambda}\ddot{\lambda}+r_{i,jk}\dot{v}_j\dot{v}_k+2r_{i,j\lambda}\dot{v}_j\dot{\lambda}+r_{i,\lambda\lambda}\dot{\lambda}\dot{\lambda}=0 \tag{4}$$

where $\mathbf{f}$: internal force vector, $\mathbf{p}_o$: generalized loading pattern, $(\;)_{,j}=\delta(\;)/\delta v_j$, $(\;)_{,jk}=\delta^2(\;)/\delta v_j\delta v_k$, $(\;)_{,\lambda}=\delta(\;)/\delta\lambda$, $(\;)_{,j\lambda}=\delta^2(\;)/\delta v_j\delta\lambda$. Summation convention will be used throughout this paper and they will run from 1 to N, where N : total number of degrees of freedom, unless otherwise specified. Making use of matrix notation, eq.(3) could be rewritten as

$$K\dot{v}-\dot{\lambda}p_o=0 \tag{5}$$

where $\mathbf{K}$ = N x N tangent stiffness matrix($K_{ij}=r_{i,j}$) and $\mathbf{p}_o=-\mathbf{r}_{i,\lambda}$. In order that eq.(5) has at least one solution, the following

$$[I-KK^-]\dot{\lambda}p_o=0 \quad \text{or} \quad [I-KK^-]^T\dot{\lambda}p_o=0 \tag{6}$$

must be satisfied[see Chapter 8 of ref.20], where $\mathbf{I}$: identity matrix and $\mathbf{K}^-$: Moore–Penrose generalized inverse matrix for $\mathbf{K}$. From eq.(6), it could be seen that the following two conditions are possible

$$\begin{aligned} &\dot{\lambda}=0 \\ &[I-KK^-]^Tp_o=0 \end{aligned} \tag{7}$$

which correspond to the condition for the occurence of limit point and bifurcation point respectively. When eq.(6) is satisfied, the solution for eq.(5) will be given as follow

$$\dot{v}=\dot{\lambda}K^-p_o+[I-K^-K]\dot{\alpha} \tag{8}$$

where $\dot{\alpha}$: arbitrary vector representing the scaling factors for the linearly independent column vectors of matrix $[\mathbf{I}-\mathbf{K}^-\mathbf{K}]$. At distinct bifurcation point, rank $\mathbf{K}$ = N–1 and rank $[\mathbf{I}-\mathbf{K}^-\mathbf{K}]=1$. This means that there is only one linearly independent column vector in matrix $[\mathbf{I}-\mathbf{K}^-\mathbf{K}]$. Therefore, eq.(8) will become

$$\dot{v}=\dot{\lambda}K^-p_o+\dot{\alpha}a \tag{9}$$

where $\mathbf{a}$: linearly independent vector of $[\mathbf{I}-\mathbf{K}^-\mathbf{K}]$. Next eq.(4) will be used in order to determine the ratio between the two scaling factors $\dot{\alpha}$ and $\dot{\lambda}$ in eq.(9). Subsitution of eq.(9) and eq.(7) into the condition for the existence of solution of eq.(4) will yield

$$[I-KK^-]^Th=0 \tag{10}$$

where $\mathbf{h}=\{h_i\}^T$ and $h_i=r_{i,jk}\dot{v}_j\dot{v}_k$. For the case of distinct bifurcation point, eq.10 will be reduced to a quadratic equation as follow

$$A\kappa^2+2B\kappa+C=0 \tag{11}$$

where $\kappa=\dot{\lambda}/\dot{\alpha}$ and A,B and C : computed constants. Using the suitable value of κ_i(i=1,2) determined from eq.(11), $\Delta\eta\dot{x}_{II}$ in eq.(1) will take the following form

$$\Delta\eta\dot{x}_{II}=(\kappa\{\, K^-p_o\, ,1\, \}^T+\{\, a\, ,0\, \}^T)\dot{\alpha} \tag{12}$$

By assigning appropriate value to $\dot{\alpha}$ in eq.(12), initial approximation for a point on bifurcation path could then be obtained.

2.2 Path-switching strategy by using line search[refs 1,2]

In this strategy, a line search is carried out on a constant loading plane introduced in the vicinity of bifurcation point along the direction of a tangent vector t_A of curve Ω(Fig.2). Curve Ω is defined by replacing equation $r_d=0(d=1$ to $N)$ with $r_{N+1}(=\lambda-\lambda_A)=0$. The purpose of line search is to locate the point D' which lies sufficiently close to point D(Fig.2). Position of point D' given as follow

$$X_{D'}=X_A+\zeta_{D'}t_A \tag{13}$$

where $X_{D'},X_A$: position vector of point D' and A respectively and $\zeta_{D'}$: distance of point D' measuring from point A, corresponds to the stationary point of total potential Π along search direction vector t_A. Iteration is then carried out on the constant loading plane in order to obtain point D by using eq.(13) as the initial approximation.

3 PATH-SWITCHING STRATEGY BY COMBINING THE USE OF GENERALIZED INVERSE AND LINE SEARCH

By using generalized inverse, one obvious advantage is that numerical trouble associated with the near-critical condition of **K** when calculating near bifurcation point could be avoided. Consequently a very reliable switching-direction predictor established in the very vicinity of bifurcation point could be obtained. By assigning suitable magnitude to this predictor, a point on the bifurcation path could then be determined. Here lies the possible source of numerical trouble as there exists no criteria with which selection of $\dot{\alpha}$ in eq.(12) could be based. The choice of $\dot{\alpha}$ has to be made on a trial-and-error basis. On the other hand, although the magnitude $\zeta_{D'}$ of switching-direction predictor t_A could be obtained through computation when line search approach is used, there exists an uncertainty in the selection of search direction vector t_A.

In this paper, the simplicity of the former approach in calculating a reliable switching-direction predictor and the characteristic of the latter approach in obtaining an initial approximation for a point on bifurcation path by searching for a stationary point of Π are combined to form an alternative path-switching strategy as described in the following paragraph.

The switching-direction predictor Δx_s(Fig.3) extending from bifurcation point B is first computed through the use of generalized inverse approach, i.e

$$\Delta x_s=(\kappa\{K^-p_o,1\}^T+\{a,0\}^T)\dot{\alpha} \tag{14}$$

κ in eq.(14) will be computed by using eq.(11). The still undetermined $\dot{\alpha}$ is then computed by first extending Δx_s until it intersects with point D' which is located on a constant loading plane introduced in the vicinity of point B(Fig.3). Since $\kappa=\lambda/\dot{\alpha}$ and by approximating $\lambda=\lambda_A-\lambda_B$, $\dot{\alpha}$ could be computed as

$$\dot{\alpha}=\Delta\lambda_{BA}/\kappa\ ,\ \kappa\neq0,\ \ \Delta\lambda_{BA}=\lambda_A-\lambda_B \tag{15}$$

In cases where $\kappa=0$ which corresponds to symmetric bifurcation point, $\dot{\alpha}$ could be specified to be the same as the size of path parameter used just prior to reaching bifurcation point during the path tracing process. Using point D' as the first approximation for a point lying on the bifurcation path, a search for a better approximation D'' is then initiated by carrying out line search on the constant loading plane. In this paper, instead of t_A(Fig.2), direction corresponding to the maximum gradient of total potential at point D'(i.e. $t_{D'}$ in Fig.3) is used as search direction. The maximum gradient of total potential at point D' will be given by the residual vector $r_{D'}$ of eq.(2). Hence search direction vector $t_{D'}$ could be evaluated as follow

$$t_{D'}=\frac{r_{D'}}{\|r_{D'}\|} \tag{16}$$

During line search, a series of points Q lying along the direction of $t_{D'}$ computed as follow

$$X_Q=X_{D'}+\zeta_Q\ t_{D'} \tag{17}$$

where ζ_Q : distance of point Q measuring from point D' along the direction $t_{D'}$, will be examined in turn in order to obtain point D". As a criteria for judging the encounter of point D"(Fig.3), the stationary condition of Π along $t_{D'}$ is used. The derivatives of Π along $t_{D'}$ at point Q could be written as follow[see ref.1]

$$\frac{d\Pi}{d\zeta}=t_{D'}^T r_Q \tag{18}$$

During line search, the vanishing of $t_{D'}^T r_Q$ is used as the criteria for terminating the search. The detected point D" could then be used as the approximation for a point on bifurcation path.

4 NUMERICAL EXAMPLE

A 3D reticulated truss dome(Fig.4) has been analysed using the proposed strategy. The occurence of bifurcation point is judged by monitoring the change in the number of negative diagonal elements of diagonal matrix **D** which is obtained during the triangulation of **K**, i.e. $\mathbf{K}=\mathbf{LDL}^T$, where **L** : a lower triangular matrix. The equilibrium path is traced using the modified Riks' normal plane-constraint arc-length method[ref.21].

Three bifurcation points B_1, B_2 and B_3 are detected before the encounter of the first limit point L_1(Fig.5). Since the number of negative diagonal elements changes in the sequence of 0-1-2-3 when traversing the three bifurcation points, all three of them are distinct bifurcation points. By checking the computed values of κ_1 and κ_2(Table 1), it can be concluded that both B_1 and B_3 are symmetric bifurcation points whilst B_2 is an asymmetric one. Table 1 lists the values of ζ where stationary point of Π are encountered in all three attempts of line search. Using the detected point D" as first approximation, bifurcation path associated with B_1 and B_2 could be obtained successfully. Tracing by using point D" detected for the case of B_3 is however not successful and the path obtained is the same as the bifurcation path associated with B_2. This failure is possibly due to reason that both B_2 and B_3 are very closely spaced as can be judged from their respective loading parameter(Table 1). Path-switching in the case of such type of bifurcation point requires further research. Fig.6 shows an example of the variation of $d\Pi/d\zeta$ along search direction $t_{D'}$ for line search carried out in the vicinity of B_1.

5 CONCLUSIONS

A path-switching strategy by combining the use of generalized inverse and line search has been proposed and tested on a 3D reticulated truss dome where three bifurcation points have been detected. The obtained numerical results show that the strategy works well in the case where the bifurcation point is distinct and isolated. Both symmetric and asymmetric bifurcation point could be dealt with successfully. Cases where bifurcation points are closely spaced together will need further research. Applicability of the strategy to other types of structures such as rigid frame, plate, shell etc. will also need further investigation.

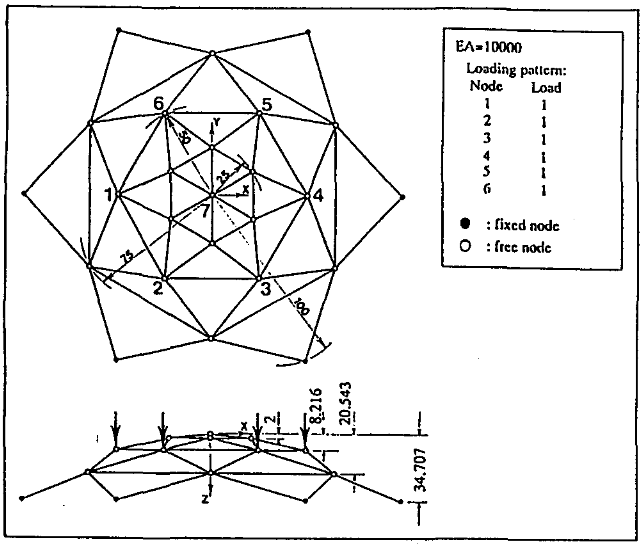
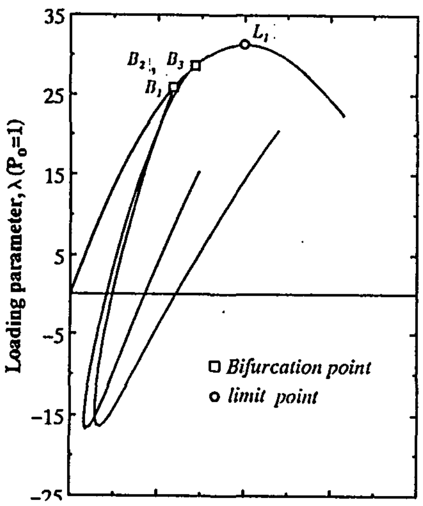


Fig. 4 A reticulated truss dome



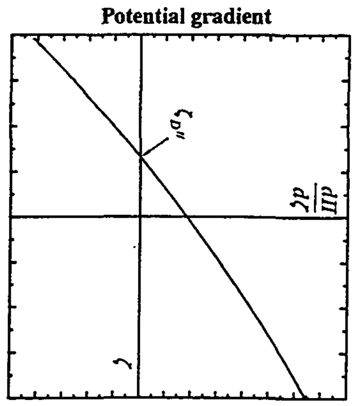


Fig. 6 Variation of $d\pi/d\xi$ along $t_{D'}$ during line search in the vicinity of point B_1

Fig. 5 Load-deflection curves for the reticulated truss dome (w:vertical displacement of node 7)

Table 1 Result of path-swiching

B.P.	κ_1	κ_2	λ_A	λ_B	ζ	α
B_1	402519	0.000	25.937	25.957	-0.658	0.050
B_2	-82.354	0.928	28.695	28.718	-0.908	-0.025
B_3	∞	0.000	28.745	28.770	-0.129	0.050

Note: 1.B.P.=Bifurcation point
2.∞=very big value

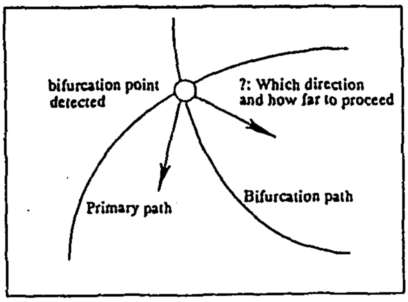


Fig. 1 Swiching from primary to bifucation path

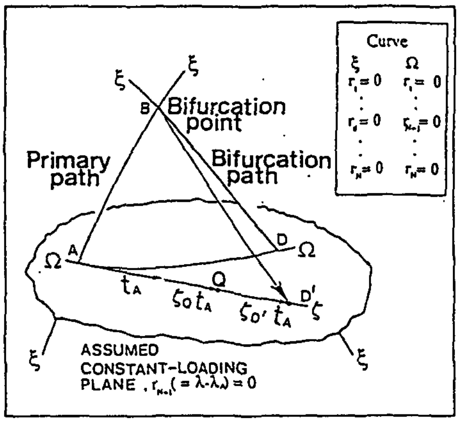


Fig. 2 Concept line search

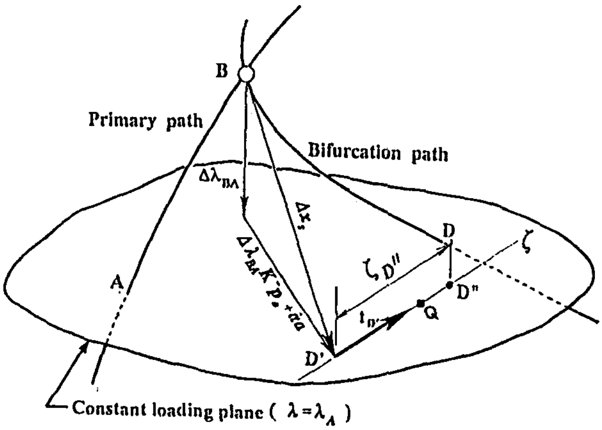


Fig. 3(a) The concept of the proposed path-swiching strategy

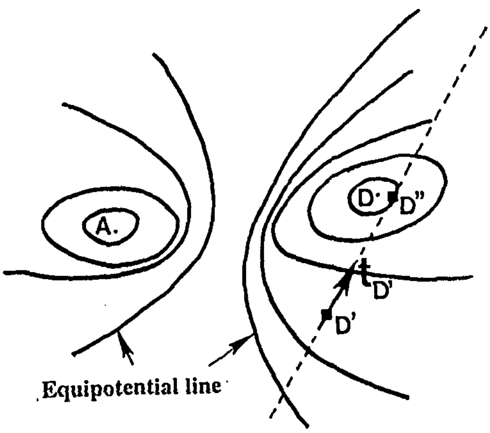


Fig. 3(b) The search for stationary point of total potential D″ in the direction of $t_{D'}$

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