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Approximate Wave Functions of Dynamic Infinite Elements for Multi-layered Halfspaces

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This paper presents a systematic procedure to obtain shape functions of the infinite elements for soil-structure interaction analysis. The function spaces are derived from the analytical solutions and appropriate assumptions based on physical interpretation. The function spaces are complete for the surface wave components, but approximate for the body wave components. Three different infinite elements are developed by using the wave functions of the derived function spaces. Numerical example analysis is presented for demonstrating the effectiveness of the present infinite elements.

1. GOVERNING EQUATION AND ANALYTICAL SOLUTIONS

Referring to *Figure 1,* the harmonic motion of a multi-layered isotropic elastic exterior region Ω_E in a soil-structure interaction system can be represented as the Navier's equation

$$(\lambda+2\mu)\nabla\nabla\cdot\mathbf{u}-\mu\nabla\times\nabla\times\mathbf{u}+\rho\omega^2\mathbf{u}=\mathbf{0} \qquad \text{in } \Omega_E \tag{1}$$

where the displacement field is defined as $\mathbf{u}(\mathbf{x};\omega)e^{i\omega t}$; ω is the frequency; $i=\sqrt{-1}$; $\mathbf{x}$ is the position vector; λ and μ are Lame's constants; and ρ is the mass density. The above equation is subjected to boundary conditions as : $\mathbf{t}_u(\mathbf{x};\omega)=\mathbf{0}$ on Γ_{f2} and $\mathbf{u}(\mathbf{x};\omega)=\bar{\mathbf{u}}(\mathbf{x};\omega)$ on Γ_I, where $\mathbf{t}_u$ is the traction vector associated with the displacement field $\mathbf{u}$, and $\bar{\mathbf{u}}$ is the displacement on Γ_I. Since this boundary-value problem, except in special cases with a bedrock, remains unsolved, approximate solutions of the free-field problem have been utilized for shape functions of the infinite elements [1,2].

Let's consider a free-field problem with the same horizontal layers and the halfspace. The solutions of this problem $\mathbf{v}(\mathbf{x};\omega)e^{i\omega t}$ lead to the transcendental equations in terms of the wavenumber k, which are related to surface waves [3]. When the displacement field $\mathbf{v}(\mathbf{x};\omega)$ in cylindrical coordinates is expressed in terms of Fourier components with respect to the azimuth, the solution $\mathbf{v}_m^{(l)}$ in the l-th layer, corresponding to the m-th wavenumber k_m and n-th Fourier component, can be obtained as

$$\mathbf{v}_m^{(l)}(\mathbf{x};\omega,k_m)=\mathbf{R}(r;\omega,k_m)\mathbf{Z}^{(l)}(z;\omega,k_m)\mathbf{s}^{(l)}(\omega) \tag{2}$$

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in which $\mathbf{R}$ is a function matrix containing the functions as $H_{n-1}^{(2)}(k_m r)$, $H_n^{(2)}(k_m r)$, $H_{n+1}^{(2)}(k_m r)$; $H_n^{(2)}$ is the second kind of Hankel function of order n; $\mathbf{Z}^{(l)}$ is a function matrix consisting of the functions as $e^{-\nu_{sl}|z|}$, $e^{-\nu_{pl}|z|}$, $e^{\nu_{sl}|z|}$, $e^{\nu_{pl}|z|}$; $\nu_{sl} = (k_m{}^2 - k_{sl}{}^2)^{1/2}$, $\nu_{pl} = (k_m{}^2 - k_{pl}{}^2)^{1/2}$, $k_{sl} = \omega / C_{sl}$, $k_{pl} = \omega / C_{pl}$, $C_{sl} = (\mu_l / \rho_l)^{1/2}$, $C_{sl} = \{(\lambda_l + 2\mu_l) / \rho_l\}^{1/2}$; and $\mathbf{s}^{(l)}(\omega)$ is a unknown coefficient vector. The functions $e^{\nu_{sl}|z|}$ and $e^{\nu_{pl}|z|}$ are to be excluded in the solution of the halfspace to satisfy the radiation condition. Now one can obtain a complete function space $\mathbf{V}_S^{(l)}$ for the surface wave components in the l-th layer of a layered medium as : $\mathbf{V}_S^{(l)} \equiv \{\mathbf{v}_m^{(l)}(\mathbf{x};\omega,k_m)\}_{m=1}^{\infty}$, in which $\{\mathbf{v}_m\}_{m=1}^{M}$ denotes an M-dimensional function space spanned over the functions $\mathbf{v}_m$ $(m = 1,2,\cdots,M)$.

Thus the displacement component $\mathbf{u}_S^{(l)}$, associated with the surface waves included in the displacement field $\mathbf{u}$, can be expanded using the functions in $\mathbf{V}_S^{(l)}$, i.e., $\mathbf{u}_S^{(l)}(\mathbf{x};\omega) = \sum_{m=1}^{\infty} \alpha_m^{(l)}(\omega)\,\mathbf{v}_m^{(l)}(\mathbf{x};\omega,k_m)$, where $\alpha_m^{(l)}(\omega)$'s are coefficients to be determined from the boundary condition on Γ_I. The displacement component $\mathbf{u}_S^{(l)}$ satisfies the field equation and the boundary condition on the free surface Γ_{f2}. However, it does not generally satisfy the boundary condition on Γ_I, because the body wave components may be included in the displacement field $\mathbf{u}$.

The displacement component $\mathbf{u}_B$ corresponding to the body waves may be represented as $\mathbf{u}_B \equiv \mathbf{u} - \mathbf{u}_S$. When a layered medium is disturbed by a vibrating foundation placed on the top surface of a layered medium, the magnitude of the component $\mathbf{u}_B$ is zero at the vertical interface while nonzero at the horizontal interface on the boundary Γ_I. It is because the inclined body waves incident upon the vertical interface are fully radiated the energy by the surface waves which may be developed by the reflections and/or the refractions of the body waves. On the other hand, when vibration sources are located in the interior region, the magnitude of the body wave component on the vertical interface is nonzero but much smaller than that on the horizontal interface. Therefore it can be interpreted that most of the body wave components $\mathbf{u}_B$ propagate through the underlying halfspace.

2. APPROXIMATE WAVE FUNCTION SPACES

In this study, the exterior domain Ω_E is divided into three domains of Ω_H, Ω_V, and Ω_C shown in *Figure 1*. The functions associated with the displacement components $\mathbf{u}_S$ are the Hankel functions of the second kind in the radial direction r and the complex exponential functions in the vertical direction z. For large value of $r\,(r \to \infty)$, one can obtain an approximation as $H_n^{(2)}(kr) \sim (2/\pi kr)^{1/2} e^{-i(kr - \pi/4 - n\pi/2)}$. Thus approximated complete function spaces for the displacement field $\mathbf{u}$ associated with the boundary condition on Γ_{f2}, can be expressed as: $\boldsymbol{P}(x) \equiv \{x^{m-1}\}_{m=1}^{N}$, $\boldsymbol{H}_S(r) \equiv \{r^{-1/2} e^{-ik_m r}\}_{m=1}^{\infty}$, and $\boldsymbol{E}_S(z) \equiv \{e^{-\nu_{sm} z}, e^{-\nu_{pm} z}\}_{m=1}^{\infty}$, where

$P(x)$ is a polynomial function space; $H_S(r)$ and $E_s(z)$ are function spaces to represent natural modes in the radial direction r and in the vertical direction z, respectively. The body wave component $\mathbf{u}_B$ in a homogeneous elastic halfspace can be represented by the second kind of spherical Hankel functions, $h_n^{(2)}(k_s R)$ and $h_n^{(2)}(k_p R)$, in which R is the distance from the vibration source to the observation point. Since the spherical Hankel function can be approximated as $h_n^{(2)}(x) \sim (1/x)\, e^{-i(x-\pi/4-n\pi/2)}$ and further as $h_n^{(2)}(x) \sim e^{-(1+i)x} e^{(\pi/4+n\pi/2)}$ for large value of x, an approximate function space for the body wave components is constructed as $H_B(x) \equiv \{e^{-(1+i)k_s x}, e^{-(1+i)k_p x}\}$ both in the radial and vertical directions,

On the basis of the function spaces described above, approximate function spaces for the given boundary-value problem have been introduced as :

$$U_H \equiv (H_S(r) \oplus H_B(r)) \otimes P(z) \tag{3}$$

$$U_V \equiv P(r) \otimes (E_S(z) \oplus H_B(z)) \tag{4}$$

$$U_C \equiv (H_S(r) \oplus H_B(r)) \otimes (E_S(z) \oplus H_B(z)) \tag{5}$$

where U_H, U_V and U_C are the respective function spaces for domains Ω_H, Ω_V, and Ω_C; $\mathbf{A} \oplus \mathbf{B}$ and $\mathbf{A} \otimes \mathbf{B}$ denote the direct sum and the tensor product of spaces $\mathbf{A}$ and $\mathbf{B}$, respectively. Proposed function spaces may be complete when the problem has a rigid bedrock. But they are complete only for the surface wave components, if the soil medium includes a underlying halfspace. One can be easily shown that the body wave components included in the approximate function spaces have finite cylindrical wavefronts similar to the interface boundary Γ_I. Note that the displacement field $\mathbf{u}$ based on the proposed function spaces enable to satisfy the compatibility conditions on the interface between the interior and the exterior domains, and also on the interfaces between two adjacent infinite elements.

3. DYNAMIC INFINITE ELEMENTS

The structure and the near field region, in this study, are modelled by using the conventional axisymmetric finite elements, and the exterior region is represented by using the proposed axisymmetric infinite elements. The cylindrical coordinate system is chosen for analyzing a layered halfspace. The mappings of the infinite element from the local coordinates to the global coordinates are defined as : $r = r_0(1+\xi)$, $z = \Sigma_{j=1}^{N} L_j(\eta)\, z_j$ for the horizontal infinite element; $r = \Sigma_{j=1}^{N} L_j(\eta)\, r_j$, $z = z_0 - \zeta$ for the vertical infinite element; $r = r_0(1+\xi)$, $z = z_0 - \zeta$ for the corner infinite element; where $L_j(\eta)$ is Legendre polynomial associated with node j; r_0 and z_0 are the coordinates of the corner point in the region Ω_C; and N is the number of nodes. The ranges of the local coordinate are $\eta \in [-1,1]$, $\xi \in [0,\infty)$ and $\zeta \in [0,\infty)$.

The displacement field in an infinite element has been easily obtained by using the proposed function spaces as

$$\mathbf{u}(r,z;\omega) = \sum_{j=1}^{N} \sum_{m=1}^{M} N_{jm}(r,z;\omega)\, \mathbf{y}_{jm}(\omega) \tag{6}$$

where $\mathbf{u}(r,z;\omega)$ is the vector of the displacement field $(=<u_r,u_\theta,u_z>^T)$; $N_{jm}(r,z;\omega)$ denotes the shape function as $L_j(\eta)f_m(\xi;\omega)$ for horizontal infinite element, $L_j(\eta)g_m(\zeta;\omega)$ for vertical infinite element, and $f_j(\xi;\omega)g_m(\zeta;\omega)$ for corner infinite element; $\mathbf{y}_{jm}(\omega)$ is the parameter vector associated with N_{jm}; $f_m(\xi;\omega)$ is in $H_S(r(\xi))\oplus H_B(r(\xi))$; $g_m(\zeta;\omega)$ is in $E_S(\zeta)\oplus H_B(\zeta)$; N is the number of nodes for horizontal and vertical infinite elements, while the number of wave functions for the corner infinite element; and M is the number of wave functions included in the displacement approximation in an infinite element. The wavenumber, $k_m(\omega)$, is a complex quantity whose imaginary part is zero or negative so that the radiation condition at infinity may be satisfied. The Equation (6) can be expressed in matrix form as

$$\mathbf{u}(r,z;\omega)=\mathbf{N}(r,z;\omega)\,\mathbf{p}(\omega) \tag{7}$$

where $\mathbf{N}(r,z;\omega)$ is the matrix of the shape functions and $\mathbf{p}(\omega)$ is the corresponding generalized coordinate vector as in Equations (8) and (9) :

$$\mathbf{N}=[\ N_{11}\mathbf{I},\cdots,N_{1M}\mathbf{I}\ |\ N_{21}\mathbf{I},\cdots,N_{2M}\mathbf{I}\ |\ \cdots\ |\ N_{N1}\mathbf{I},\cdots,N_{NM}\mathbf{I}\ ] \tag{8}$$

$$\mathbf{p}=\left\langle\ \mathbf{y}_{11}{}^T,\cdots,\mathbf{y}_{1M}{}^T\ |\ \mathbf{y}_{21}{}^T,\cdots,\mathbf{y}_{2M}{}^T\ |\ \cdots\ |\ \mathbf{y}_{N1}{}^T,\cdots,\mathbf{y}_{NM}{}^T\ \right\rangle^T \tag{9}$$

in which $\mathbf{I}$ is a 3×3 identity matrix.

For the purpose of constructing the system matrices, it is required to express the displacement field in each infinite element in terms of shape functions associated with the nodal displacements $\mathbf{u}^{(n)}$, the displacements along the sides of the infinite element $\mathbf{u}^{(s)}$, and the internal displacements $\mathbf{u}^{(i)}$. Hence, Equation (7) is rewritten as

$$\mathbf{u}=\mathbf{u}^{(n)}+\mathbf{u}^{(s)}+\mathbf{u}^{(i)}\qquad\text{or}\qquad\mathbf{u}(r,z;\omega)=\tilde{\mathbf{N}}(r,z;\omega)\,\mathbf{q}(\omega) \tag{10}$$

where the respective expressions for the horizontal, vertical and corner infinite elements are :

$$\mathbf{u}^{(n)}=\sum_{j=1}^{N}L_jf_1\mathbf{d}_j,\quad \mathbf{u}^{(s)}=\sum_{l=2}^{N_h}L_1\,\phi_l\,\mathbf{a}_{1l}+\sum_{l=2}^{N_h}L_N\,\phi_l\,\mathbf{a}_{Nl},\quad \mathbf{u}^{(i)}=\sum_{j=2}^{N-1}\sum_{l=2}^{N_h}L_j\,\phi_l\,\mathbf{a}_{jl} \tag{11}$$

$$\mathbf{u}^{(n)}=\sum_{j=1}^{N}L_jg_1\mathbf{d}_j,\quad \mathbf{u}^{(s)}=\sum_{m=2}^{N_v}L_1\,\psi_m\,\mathbf{b}_{1m}+\sum_{m=2}^{N_v}L_N\,\psi_m\,\mathbf{b}_{Nm},\quad \mathbf{u}^{(i)}=\sum_{j=2}^{N-1}\sum_{m=2}^{N_v}L_j\,\psi_m\,\mathbf{b}_{jm} \tag{12}$$

$$\mathbf{u}^{(n)}=f_1g_1\mathbf{d}_1,\qquad \mathbf{u}^{(s)}=\sum_{l=2}^{N_h}\phi_l\,g_1\,\mathbf{a}_{1l}+\sum_{m=2}^{N_v}f_1\,\psi_m\,\mathbf{b}_{Nm},\qquad \mathbf{u}^{(i)}=\sum_{l=2}^{N_h}\sum_{m=2}^{N_v}\phi_l\,\psi_m\,\mathbf{c}_{lm} \tag{13}$$

in which $\phi_l(\xi;\omega)$ and $\psi_m(\zeta;\omega)$ denote the wave functions as $f_l(\xi;\omega)-f_1(\xi;\omega)$ and $g_m(\zeta;\omega)-g_1(\zeta;\omega)$, respectively; N_h and N_v are the numbers of horizontal and vertical wave components used in the infinite elements, respectively; $\mathbf{d}_j(\omega)$ is the displacement vector at node j; and $\mathbf{a}_{jl}(\omega)$, $\mathbf{b}_{jm}(\omega)$ and $\mathbf{c}_{lm}(\omega)$ are the respective parameter vectors associated with the shape functions $L_j(\eta)\phi_l(\xi;\omega)$, $L_j(\eta)\psi_m(\zeta;\omega)$ and $\phi_l(\xi;\omega)\psi_m(\zeta;\omega)$. $\tilde{\mathbf{N}}(r,z;\omega)$ is the new shape function matrix and $\mathbf{q}(\omega)$ is the new generalized coordinate vector.

The relationship between two generalized coordinates $\mathbf{p}(\omega)$ and $\mathbf{q}(\omega)$ can be obtained as $\mathbf{p}(\omega) = \mathbf{T}\,\mathbf{q}(\omega)$, where $\mathbf{T}$ is the transformation matrix that can be easily derived from Equations (7) and (10). Thus one can obtain the relationship between two shape function matrices $\tilde{\mathbf{N}}(r,z;\omega) = \mathbf{N}(r,z;\omega)\,\mathbf{T}$.

To construct the stiffness and mass matrices of an infinite element, it is required to perform numerical integrations of complex exponential functions in the infinite direction. For the computational efficiency, at first, the element matrices are computed for the old coordinates $\mathbf{p}(\omega)$ by using the Gauss-Laguerre quadrature with complex coefficients[1], then the results are transformed into those for the new coordinates $\mathbf{q}(\omega)$ as: $\mathbf{K}_{qq} = \mathbf{T}^T\mathbf{K}_{pp}\mathbf{T}$ and $\mathbf{M}_{qq} = \mathbf{T}^T\mathbf{M}_{pp}\mathbf{T}$. Prior to assembling the element matrices into the system matrices, the degrees of freedom associated with the internal displacement $\mathbf{u}^{(i)}$ in the element matrices are condensed out.

4. NUMERICAL EXAMPLES AND DISCUSSIONS

The infinite elements have been used to obtain the horizontal and rocking impedance functions for a rigid disk placed on layered halfspace as shown in *Figure 2*. Both the horizontal layer and the halfspace are assumed to be elastic, homogeneous and isotropic with different shear velocities (C_{s1} and C_{s2}), densities (ρ_1 and ρ_2) and Poisson's ratios (ν_1 and ν_2). Example analysis is carried out for the case with values : $C_{s1}/C_{s2} = 0.8$, $\rho_1/\rho_2 = 0.85$, $\nu_1 = \nu_2 = 0.25$.

The impedance functions of the disk have been obtained and are expressed in terms of the dimensionless frequency a_0 $(=\omega R_0/C_{s1})$. These results are found to be in good agreements with the results obtained by Yang and Yun[1] and Luco[4]. In the Yang and Yun's study, the underlying halfspace has been modelled using the finite elements for a hemisphere with the radius r_0 and the radiational infinite elements for the remaining region. Numerical results indicate that the present method gives slightly stiffer solutions than those by other methods but it is easier to model the layered halfspace and requires less number of the degrees of freedom of the soil-structure.

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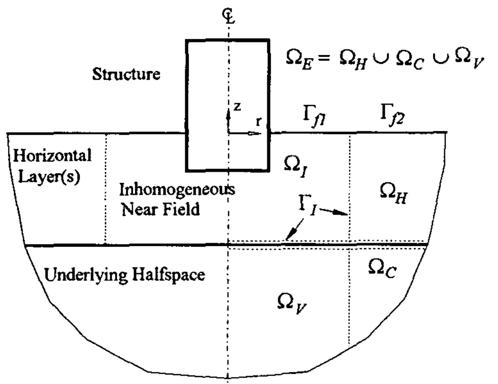

Figure 1. An idealized SSI system

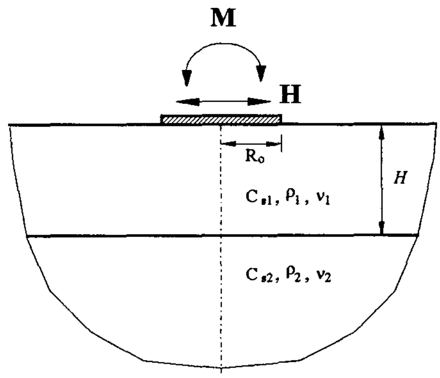

Figure 2. Rigid disk on layered halfspace

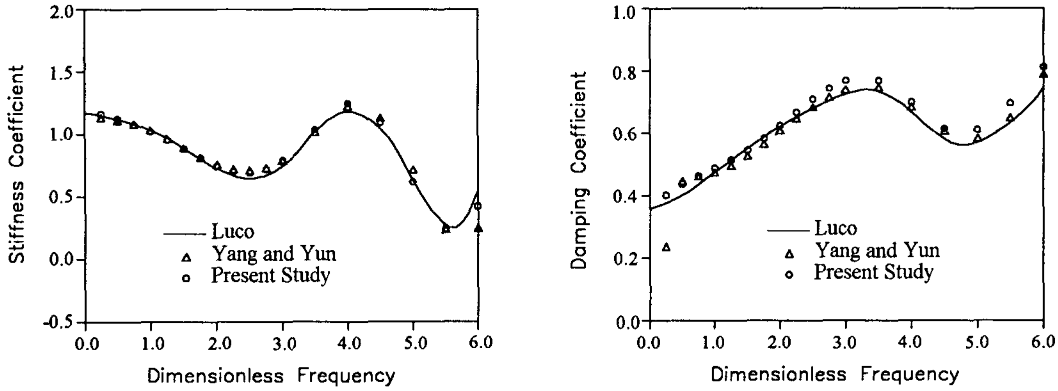

Figure 3. Horizontal impedance function of rigid disk on layered halfspace ($H / R_0 = 1.0$)

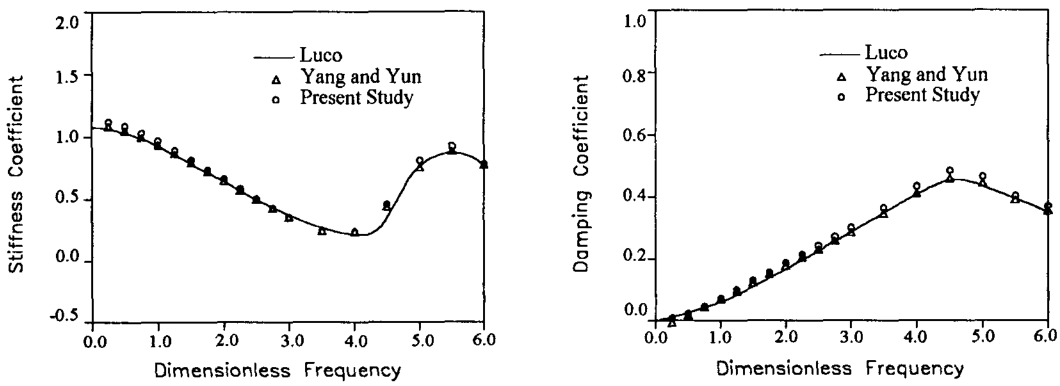

Figure 4. Rocking impedance function of rigid disk on layered halfspace ($H / R_0 = 1.0$)