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p-Version Finite Element Model of Cracked Plates Including Shear Deformation under Flexural Behavior

휨거동을 받는 균열판의 전단변형을 고려한 p-Version 유한요소모델

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ABSTRACT

The new p-version crack model is proposed to estimate the bending stress intensity factors of the thick cracked plate under flexure. The proposed model is based on high order theory and C^0-plate element including shear deformation. The displacements fields are defined by integrals of Legendre polynomials which can be classified into three groups such as basic mode, side mode and internal mode. The computer implementation allows arbitrary variations of p-level up to a maximum value of 10. The bending stress intensity factors are computed by virtual crack extention approach. The effects of ratios of thickness to crack length(h/a), crack length to width(a/W) and boundary conditions are investigated. Very good agreement with the existing solution in the literature are shown for the uncracked plate as well as the cracked plate.

1. INTRODUCTION

One of the basic requirements for the strength analysis of plates containing flaws or cracks is the knowledge of the singular character of the stress field in the neighborhood of the point at the crack tip. The stress intensity factors which are in fact the strength of the stress singularities at crack tips are applied to predict the static strength of cracked bodies by Irwin[1] for plane extension, symmetric with respect to the crack. The stress intensity factors have been shown to control the rate of crack propagation under cyclic loading in such situations. Several investigators have discussed the nature of the local stresses around a sharp crack in a thin plate subjected to out-of-plane bending loads. Based on the Poisson-Kirchhoff theory of thin plate and the technique of Fadle eigenfunction expansion, Williams[2] found that the elastic bending stresses near the tip of a semi-infinite crack vary as the inverse square root of the radial distance from the crack front. His results were not complete in that the strength or magnitude of the local stresses was left undetermined. Sih *et. al.*[3] cleared the way for finding the coefficients in the eigen function exoansions by application of the theory of complex functions. However, the results in [2-3] were obtained from the classical fourth-order theory of thin plates, the edge conditions at the crack surfaces are satisfied only in an approximate manner in that the three physically natural boundary conditions of prescribing bending moment, twisting moment, and transverse shear stress are replace by two conditions. Owing to such a replacement, the stress distribution in the immediate neighborhood of the crack edges will naturally be affected and will not be accurate.

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To overcome that aforementioned shortcoming, Knowles and Wang[4] used Reissner's sixth-order plate theory to satisfy all the physically natural boundary conditions along the crack edges with the help of singular integral equations. However, the extension of the method to cracked plates with finite plate thickness does not appear to be tractable. For the examination of the effect of plate thickness, Hartranft and Sih[5] put forward higher order theories that included the effect of transverse shear deformations and essentially solved the problem of infinite plates subjected to uniform moment. Murthy *et. al.* [6] used Reissner's plate theory and applied collocation and succesive integration methods to satisfy the boundary conditions for a finite plate with a through crack.

The associated numerical difficulties led them to conclude that the finite element method is better suited for such problems. Wilson and Thompson[7] used h-version of the finite element method to compute the SIF for the through-the-thickness crack in a plate under flexure. They used a large number of 3-node triangular C^1-elements(actually 257 and 305 node models) and computed the value of SIF using the direct method. Hilton[8] used 48 twenty-noded three dimensional isoparametric elements including special singularity element(279 node model) to compute a SIF. Yagawa *et. al.*[9] used a superposition method of analytical and finite element solutions on the basis of 36 eight-noded isoparametric elements(133 node model) together with the reduced integration technique. Alwar *et. al.*[10] used 110 three dimensional quadratic isoparametric elements incluiding degenerate triangular quarter-point prism elements at the crack tip region to model one eighth of the plate and computed SIF by COD method. The results presented by earlier workers(Hilton and Yagawa) for finite plate under uniform moments differ singnificantly from the results by Alwar. Alwar indicated that the difference may be due to the coarseness of the mesh used by those. Chen *et. al.*[11] presented the hybrid-displacement singular element model based on the Kirchhoff plate theory for the bending analysis of thin cracked infinite plate and the transverse pressure analysis of thin finite cracked plate with various boundary condition. Kang *et. al.*[12] used 35 hybrid Mongrel elements, which is special crack tip element, for infinite cracked plate under bending and finite thin plate under transverse pressure.

Since the singularity at the through crack corner of finite plate subjected to various load and boundary condition is unresolved issue, the computation of SIF of plate is very interesting problem. In this paper, the LEFM analysis of plates subjected to out-of-plane loading is considered using p-version of the finite element method. The superiority of the P-version of the finite element method in LEFM computations was demonstrated by Metha[13], Basu[14], Woo[15]. It was shown by numerical experimentations and analytical proof[16] that the rate of convergence of p-extension is twice the rate of h-extension in the finite element modeling of plane stress/strain problems with sharp cracks when the number of degrees of freedom is increased by uniform or quasi-uniform mesh refinement. The objective of this syudy is to determine the SIF for finite plate with various boundary condition and to investigate the variation of the same across the plate thickness under distributed edge moments and transverse pressures by the p-version crack model based on virtual crack extension method and integrals of Legendre polynomials.

2. STRESS FIELD NEAR CRACK TIP AND STRESS INTENSITY FACTORS

Irwin[1] obtained the form of the elastic stress distribution in the vicinity of a crack tip in extensional problems. Williams[2] extended his analysis to thin plates subjected to bending out of the plane. In each case it is shown that the significant stresses in the vicinity of the crack tip are those associated with the singularity of stress of the order $r^{-1/2}$, where r is radial distance from the crack tip. Moreover, the distribution of stress in each, extension or bending, is unique; i.e., its functional form in terms of coodinates measured from the crack tip is always the same. Hence, for bending, Williams' results will be modified to define moment intensity factors in bending in a manner consistent with Irwin's definitions.

$$\sigma_r = \frac{12M_r\delta}{h^3}\left\{[\cos\frac{3\theta}{2} - \frac{3+5\nu}{7+\nu}\cos\frac{\theta}{2}]\frac{3K_I G\delta}{2} + [-\sin\frac{3\theta}{2} - \frac{3+5\nu}{5+3\nu}\sin\frac{\theta}{2}]\frac{3K_{II}G\delta}{2}\right\}$$

$$\sigma_\theta = \frac{12M_\theta\delta}{h^3}\left\{[-\cos\frac{3\theta}{2} - \frac{5+3\nu}{7+\nu}\cos\frac{\theta}{2}]\frac{3K_I G\delta}{2} + [\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}]\frac{3K_{II}G\delta}{2}\right\} \qquad (1)$$

$$\tau_{r\theta} = \frac{12M_{r\theta}\delta}{h^3}\left\{[-\sin\frac{3\theta}{2} + \frac{1-\nu}{7+\nu}\sin\frac{\theta}{2}]\frac{3K_I G\delta}{2} + [-\cos\frac{3\theta}{2} - \frac{1-\nu}{5+3\nu}\cos\frac{\theta}{2}]\frac{3K_{II}G\delta}{2}\right\}$$

where δ is a coordinate perpendicular to the middle plane of the plate, G and h are associated with shear modulus of elasticity and thickness of a plate, respectively.

When the crack is tilted at an angle β with reference to the plane about which a bending moment of magnitude M_o is applied, Sih[3] and Hartranft[5] have provided the moment intensity factors shown in Eqs.(2)-(4).

$$K_I = \Phi(1) M_o \sqrt{a} \sin^2\beta \qquad (2)$$

$$K_{II} = \Psi(1) M_o \sqrt{a} \sin\beta\cos\beta \qquad (3)$$

$$K_{III} = -\frac{\sqrt{10}}{(1+\nu)h}\Omega(1) M_o \sqrt{a} \sin\beta\cos\beta \qquad (4)$$

where the functions $\Phi(1)$, $\Psi(1)$ and $\Omega(1)$ are computed numerically from integral equations. Their values are function of $h/a\sqrt{10}$ for different Poisson's ratio. In case of mode I, the moment intensity factor K_I can be modified to the bending stress intensity factor k_1 under the plain strain condition such that

$$k_1 = \Phi(1)\sigma_b\sqrt{a} \qquad (5)$$

where $$\sigma_b = \frac{6}{h^2}M_o \qquad (6)$$

3. INTEGRAL OF LEGENDRE POLYNOMIALS

The general form of integral of Legendre polynomials are defined over the standard domain such that ;

$$F_{j+1}(\xi) = \sqrt{\frac{2j-1}{2}}\int_{-1}^{\xi} P_j(t)\,dt \qquad (7)$$

where

$$P_j(t) = \frac{1}{2^i i!}\frac{d^i}{dt^i}(t^2-1)^i \qquad (8)$$

where i = 0, 1, 2, · · · .
For a standard quadrilateral domain, with ξ, η $(-1 \leq \xi \leq 1$ and $-1 \leq \eta \leq 1)$ as the coordinate of a point, the shape functions are built from the 1-D shape functions $F_i(\xi)$ as follows [15];

1. The four vertex modes are $F_i(\xi) \cdot F_j(\eta)$ with $i, j = 1, 2$.
2. The side modes van be obtained by multiplying $F_p(\xi)$ with $(\eta+1)$ and $(\eta-1)$ for sides, $\eta = \pm 1$ and $F_p(\eta)$ with $(\xi+1)$ and $(\xi-1)$ for the sides $\xi = \pm 1$. Here $p \geq 2$ is the order of the shape function, i.e., $p=2$ indicates the quadratic shape functions.
3. The internal modes are valid for $p \geq 4$ only, and can be obtained by taking the product $F_i(\xi) \cdot F_j(\eta)$ so that $i+j=p$, and both i and j are greater than or equal to 2.

4. COMPUTATION OF STRESS INTENSITY FACTORS

The finite element method has been used by a number of investigators to determine elastic stress intensity factors for cracked bodies. The characteristic elastic square root singularity has been represented by the use of virtual crack extension method in this work. For the virtual crack extension method, it may be shown in the absence of body forces.

$$\int_0^L G(s) \cdot \delta A(s)\, ds = -\frac{1}{2} \{u\}^T \cdot \Delta[K] \cdot \{u\} \tag{9}$$

where G(s) is Griffith's energy, and a function of position s along the crack front, a is the length of the crack front, and {u} is the vector of nodal point displacements found from the finite element computation. The change in the stiffness matrix Δ[K], for a given virtual crack extension may be written as a forward difference, namely

$$\Delta[K] \cong [K]_{a+\delta a} - [K]_a \tag{10}$$

Consider a crack of length a which advances by an incremental amount δa, thereby causing a release of strain energy of amount δU. Thus the incremental crack surface $\delta A(s)$ for axisymmetric bodies is defined by;

$$\delta A(s) = h(a + \delta a) - ha \tag{11}$$

where h is the thickness of plate.

Therefore, the strain energy release rate $G(s)$ for axisymmetric cracked bodies can be expressed as;

$$G(s) = \frac{\delta U}{\delta A(s)} \tag{12}$$

then the stress intensity factors are directly related to the value of $G(s)$ caused by a crack extension in the appropriate mode. Hratranft and Sih[5] have provided k_1 expressions under plane strain condition including the effect of the plate thickness at the upper surface of the cracked plate $(\delta = h/2)$. The relationship between k_1 and G for plane strain condition is expressed by ;

$$k_1 = \sqrt{\frac{GE}{1-\nu^2}} \tag{13}$$

in which G is the strain energy release rates under mode I action, ν is Poisson's ratio and E is Young's modulus. However, since the quantity $G(s)$ is very sensitive to the crack length increment δa, the sensitivity test was investigated between $G(s)$ and δa. From this, δa was adopted by $10^{7}a$.

5. NUMERICAL RESULTS

5.1 A centrally cracked square plate under uniform edge moments

The first example is a centrally cracked square plate subjected to uniform edge normal bending moments. Because of symmetry, only a quarter of the structure is discretized into 2x2 mesh shown in Fig.1. The geometric and mechanical data of the problem are given by the length of square plate $W=10$, $E=1\times10^{6}$, $\nu=0.3$ and constant normal bending moment M_o per unit length. The p-version results of the non-dimensional bending stress intensity factor $\Phi(1)$ defined by Eq.(5) have been compared with 3-D finite element solutions by Alwar[10] and theoretical solution of the infinite cracked plates obtained by Hartranft and Sih[5] when $a/W=0.03$. It is noted that the p-version 4-elemet model with p=8, in case of the infinite plate, which is almost equal to Alwar's analysis shows a good comparision with Hartranft's theoretical solution within 4% relative errors. The non-dimensional bending stress intensity factors $\Phi(1)$ for the finite plate with respect to the ratio of a/W have been shown in Fig.2 as the ratio of $h/2a$ varies. The effects of thickness of the finite cracked plate have been tested with the same p-version model in Fig.3. The p-version finite element model is very close to 3-D analysis by Alwar[10], however, 2-D analysis by Yagawa[9] gives large difference in reference with both p-version model and Alwar's solution when $a/W=0.5$. In Fig.4, the effect of Poisson's ratio has been investigated as the thickness is increased. As we aware of it from this figure, the non-dimensional bending stress intensity factors have been affected by the variation of Poisson's ratio. Unfortunately, there are not published papers in literature up to this stage.

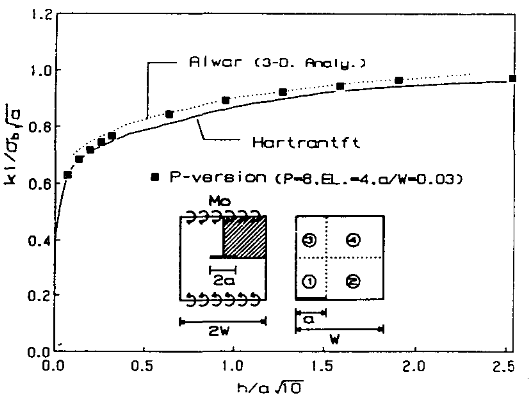

Fig.1 Variation of $\Phi(1)$ of infinite plate when a/W=0.03

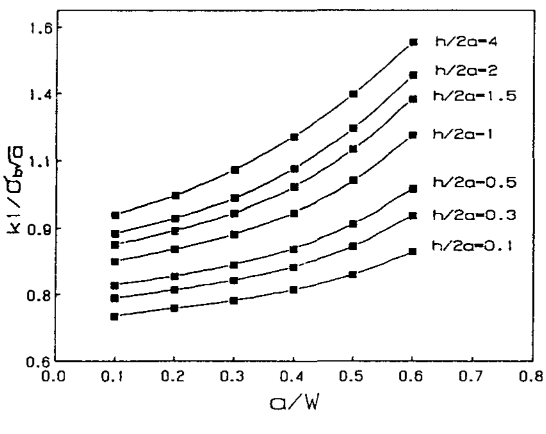

Fig.2 Variation of $\Phi(1)$ vs thickness to crack length ratio

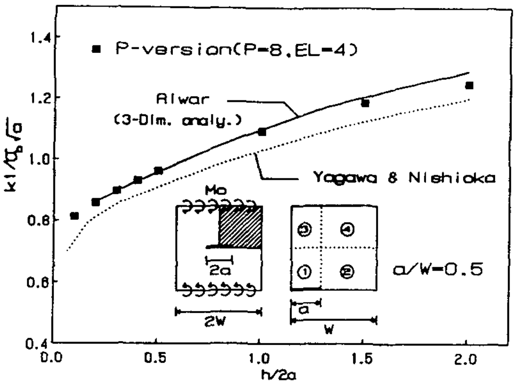

Fig.3 Comparison of Φ(1) with finite element solutions when a/W=0.5

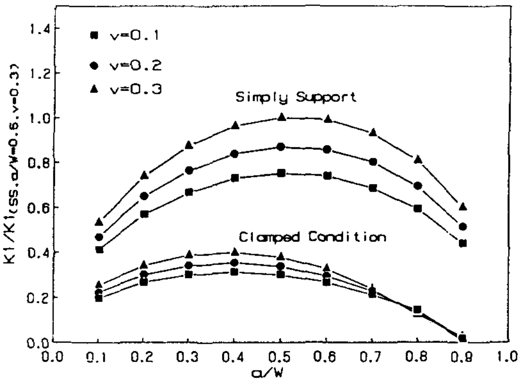

Fig.4 Effect of Poisson's ratio with thickness to crack length ratio

5.2 A centrally cracked square plate under uniform pressures

This example has been tested the effect of bending stress intensity factors with different crack length and boundary conditons in case of a centrally cracked square plate under uniform transverse loading p_o. The finite element mesh and geometric/mecanical data are the same as a centrally cracked square plate under edge moments. The bending stress intensity factors with respect to crack length for simply supported(SS) and clamped condition(CC) have been investigated in Fig.5 where the stress intensity factotrs were normalized by the reference value

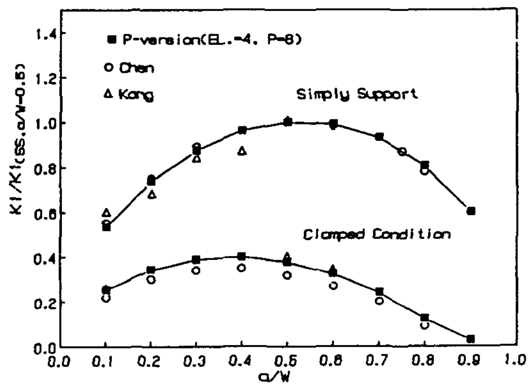

Fig.5 Normalized bending stress intensity factors vs a/W ratio.

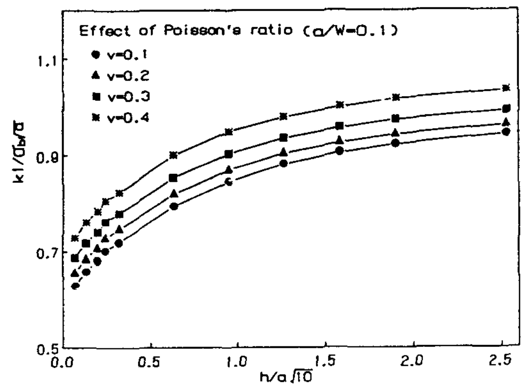

Fig.6 Variation of normalized bending stress intensity factors with a/W and Poisson's ratio.

of bending stress intensity factor of a/W=0.5 with SS condition and this problem was also solved under the condition of thin cracked plate such as W/h=30. The p-version finite element model shows a good comparison with the hybrid displacement singular element approach by Chen[11] and the hybrid mongrel element model by Kang[12], respectively. It is also noted that the bending stress intensity factor increases to maximum value at a certain ratio of a/W and decreases even when crack length is growing. As we expected, the effect of Poisson's ratio is very large as well as the a centrally cracked plate under uniform edge moments. However, it is observed that Poisson's ratio has effect on the non-dimensional bending stress intensity factor in SS boundary condition rather than in CC boundary condition. The configuration of the square cracked plate with two opposite edges simply supported and the other two edges free is shown in Fig.7 where the loading condition and mechanical properies are unchaged like previous problem. The solutions obtained by 4-element p-version model with p=8 are in good accordance with those by Sosa *et al.*[17]. In this case, the non-dimensional bending stress intensity factor grows rapidly as the crack length increases.

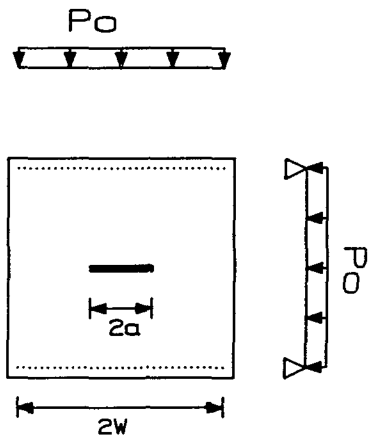

Fig.7 Configuration of cracked square plate with two opposite edges simply supported and the other edges free

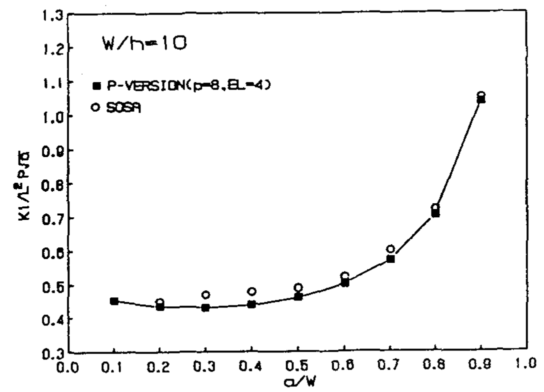

Fig.8 Growth of normalized bending stress with a/W ratio

6. SUMMARY AND CONCLUSIONS

The new p-version crack model has been proposed that is based on Reissner-Mindlin theory, integral of Legendre polynomials and virtual crack extension method. Also, this model is under the scope of theory of small scale yielding. Throughout treating examples, the cracked square plates have been investigated with respect to effects of size of finite plate, Poisson's ratio, loading conditions of uniform edge moments and transverse pressures, various boundary conditions, and variation of thickness. The physical domains are modeled by only four C^0-hierarchical plate elemens with p-level=8 to get the same levels of accuracy from the special singular finite element solutions in literatures. From this study, it is concluded that the proposed crack model shows superior performance to the exsisting crack model on the basis of hybrid singular elements in the sense of modeling simplicity, accuracy and a tremendous savings in CPU and user's time. Also, it is apparent that p-version crack model is very suitable for LEFM analysis irrespective of loadings and geometric/mechanical properties.

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