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**Abstract**

The main purpose of this paper is to introduce and develop the notion of a fuzzy measure in separable Banach space. This definition of fuzzy measure is a natural generalization of the set-valued measure. Radon-Nikodým theorems for fuzzy measure are established.

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# 1 Introduction

The concept of fuzzy set was introduced by Zadeh (1965). Subsequent developments focused on applications of this concept to pattern recognition and system analysis, among other areas.

The topic of set valued and fuzzy measures has received much attention in the last few years because of its usefulness in several applied fields like mathematical economics and optimal control. Significant contributions in this area were made by Hiai, [2], [3], Bán [1], Zhang Wen Kin, Li Teng [10]. In these works appeared several different definitions of the notion of set valued and fuzzy measure. The purpose of this paper is to give and compare different definitions of fuzzy measure whose values are fuzzy sets defined on a real separable Banach space. One of advantages of this concept of fuzzy measure is that it enables us to use the rich mathematical apparatus of the theory of set-valued measures.

# 2 Preliminary notions on set valued measures

Let $(\Omega, \mathcal{A})$ be a measurable space with $\mathcal{A}$ a $\sigma$-field of measurable subsets of the set $\Omega$. Let $(\mathcal{X}, \|\cdot\|)$ be a real separable Banach space, $\mathcal{P}(\mathcal{X})$ the family of all nonempty subsets of $\mathcal{X}$ and $\mathcal{P}_c(\mathcal{X})$ the set of all convex compact subsets of $\mathcal{X}$. If $A \in \mathcal{P}(\mathcal{X})$ the number $|A|$ is defined by

$$|A| = \sup_{x \in A} \|x\|.$$

For two closed bounded sets $A, B \in \mathcal{P}(\mathcal{X})$ denote the Hausdorff metric of $A$ and $B$,

$$H(A,B) = \max\{\sup_{x\in A} \inf_{y\in B} \|x-y\|, \sup_{y\in B} \inf_{x\in A} \|x-y\|\}.$$

A set valued, set function $M : \mathcal{A} \to \mathcal{P}(\mathcal{X})$ is said to be a set valued measure (multimeasure) if it satisfies the following two requirements:

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(i) $M(\cdot)$ is countably additive, in the sense that given any sequence $\{A_n\}_{n\in N}$ of pairwise disjoint elements of $\mathcal{A}$ we have that

$$M(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{n} M(A_n),$$

where

$$\sum_{n=1}^{\infty} M(A_n) = \{x \in \mathcal{X} : x = \sum_{n=1}^{\infty} x_n \ , \ \sum_{n=1}^{\infty} \|x_n\| < \infty \ , \ x_n \in M(A_n)\}$$

(ii) $M(\emptyset) = \{0\}$.

As for single valued measures, we have the notion of total veriation $|M|(\cdot)$ of $M(\cdot)$. For $A \in \mathcal{A}$ we define

$$|M|(A) = \sup \sum_{i=1}^{n} |M(A_i)|,$$

where the supremum is taken over all finite measurable partitions $\{A_1, \ldots, A_n\}$ of $A$. If $|M|(\Omega) < \infty$, then we say that $M(\cdot)$ is of bounded variation. It is easy to see that in this case the sums in the definition of $\sum_{n=1}^{\infty} M(A_n)$ are absolutely convergent. Finally, we say that $M(\cdot)$ is $\mu$ continuous, where $\mu$ is a single valued vector measure if and only if for any $A \in \mathcal{A}$ for which $\mu(A) = 0$ we have $M(A) = \{0\}$.

## 3 Fuzzy valued measure

Let $\mathcal{F}_c(\mathcal{X})$ be a set of fuzzy sets $u : \mathcal{X} \to [0,1]$ such that

1. $u(1) \neq \emptyset$
2. $u_\alpha \stackrel{\text{def}}{=} \{x \in \mathcal{X} : u(x) \geq \alpha\} \in \mathcal{P}_c(\mathcal{X})$ ($u_\alpha$ is $\alpha$-cut of $u$)

The mapping $\mathcal{M} : \mathcal{A} \to \mathcal{F}_c(\mathcal{X})$ is a fuzzy valued measure if it satisfies

1. $\mathcal{M}(\emptyset) = I_{\{0\}}$ where $I_{\{0\}}$ is indicator function of $\{0\}$
2. if $\{A_n\} \subset \mathcal{A}, A_i \cap A_j = \emptyset, i \neq j$
   $\mathcal{M}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathcal{M}(A_n)$,
   where

$$\sum_{n=1}^{\infty} \mathcal{M}(A_n)(x) = \sup\{\bigwedge_{n=1}^{\infty} \mathcal{M}(A_n)(x_n), x = \sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} \|x_n\| < \infty\}$$

**Theorem 1** *If $\mathcal{M} : \mathcal{A} \to \mathcal{F}_c(\mathcal{X})$ is a fuzzy valued measure, then $\mathcal{M}_\alpha : \mathcal{A} \to P_c(\mathcal{X})$, where*

$$\mathcal{M}_\alpha(A) = \{x \in \mathcal{X} : \mathcal{M}(A)(x) \geq \alpha\},$$

*is a set valued measure for all $\alpha \in (0,1]$.*

**Theorem 2** *Let $\mathcal{M}_\alpha, \alpha \subset (0,1]$, be a family of compact, convex set valued measures and let $\mathcal{M}_\beta(A) \subseteq \mathcal{M}_\alpha(A)$ for all $0 < \alpha < \beta \leq 1$ and all $A \subset \mathcal{A}$. Then $\mathcal{M}(A)(x) = \sup_{\alpha\in(0,1]}\{x \in \mathcal{M}_\alpha(A)\}$ is a fuzzy valued measure.*

## 4 Radon Nikodým derivative

Hereafter, let $(\Omega, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space. Let $L(\Omega, \mathcal{A}, \mu, \mathcal{X}) = L$ denotes the Banach space of measurable functions $f : \Omega \to \mathcal{X}$ such that the norm $\|f\|_L = \int_\Omega \|f(\omega)\| d\mu$ is finite and let $h$ be a Hausdorff metric in $L$.

If $F : \Omega \to \mathcal{P}_c(\mathcal{X})$ is a measurable multivalued function, then a mesurable function $f : \Omega \to \mathcal{X}$ is called a measurable selection of $F$ if $f(\omega) \in F(\omega)$ for all $\omega \subset \Omega$ and

$$S_F = \{f \in L : f(\omega) \in F(\omega)\ \mu - a.e.\}.$$

The set $S_F$ is a closed subset of $L$. A multivalued function $F : \Omega \to \mathcal{P}_c(\mathcal{X})$ is called integrably bounded if there exists $\mu$- integrable function $s : \Omega \to R$ such that $\sup_{x \in F(\omega)} \|x\| \leq s(\omega)$ for all $\omega \in \Omega$. $F$ is integrably bounded if and only if $S_F$ is nonempty and bounded.

Let $\mathcal{L}(\Omega, \mathcal{A}, \mu, \mathcal{X}) = \mathcal{L}$ denote the space of all integrably bounded multivalued functions.

If $F$ is a measurable multivalued function $F : \Omega \to \mathcal{P}_c(\mathcal{X})$, then the integral of $F$ is defined by

$$\int_\Omega F d\mu = \{\int_\Omega f d\mu : f \in S_F\}.$$

A fuzzy random variable is a function $X : \Omega \to \mathcal{F}_c(\mathcal{X})$, where $\mathcal{F}_c(\mathcal{X})$ denotes all functions (fuzzy sets of $\mathcal{X}$) $X(\cdot) : \mathcal{X} \to [0,1]$ such that

1. $\{x \in \mathcal{X} : X(\omega)(x) \geq \alpha\} = X_\alpha(\omega)$ is nonempty, convex and compact for all $x \in (0,1]$.
2. $X_\alpha : \Omega \to \mathcal{F}(\mathcal{X})$, is a measurable function for all $\alpha \in (0,1]$.

A fuzzy random variable $X$ is called integrably bounded if $X_\alpha$ is integrably bounded for all $\alpha \in (0,1]$, i.e. of for all $\alpha \in (0,1]$, $X_\alpha \in \mathcal{L}$.

Let $\Lambda(\Omega, \mathcal{A}, \mu, \mathcal{X}) = \Lambda$ be the set of all integrably bounded fuzzy random variables.

The integral $\int_\Omega X d\mu$ of $X \in \Lambda$ is introduced by

$$(\int_\Omega X d\mu)_\alpha = \int_\Omega X_\alpha d\mu \quad , \quad \alpha \in (0,1].$$

and

$$(\int_\Omega X d\mu)(x) = \sup_{\alpha \in (0,1]} \{x \in \int_\Omega X_\alpha d\mu\}.$$

If $A \subset \mathcal{A}$, $\int_A X d\mu$ is the integral of the restriction $X|A$.

A fuzzy valued measure $\mathcal{M} : \mathcal{A} \to \mathcal{F}_c(\mathcal{X})$ is of bounded variation if $\mathcal{M}_\alpha : \mathcal{A} \to \mathcal{P}_c(\mathcal{X})$ is a set valued measure of bounded variation for all $\alpha \in (0,1]$ and $\mathcal{M}$ is $\mu$-continuous if $A \in \mathcal{A}$ with $\mu(A) = 0$ implies that $\mathcal{M}(A) = I_{\{0\}}$ where $I_{\{0\}}$ is the indicator function of $\{0\}$.

**Theorem 3** *If $X \in \Lambda$, then*

$$\mathcal{M}(A) = \int_A X d\mu$$

*is a $\mu$-continuous fuzzy measure of bounded variation.*

**Theorem 4** *If $\mathcal{X}$ has the Radon-Nikodým property, $\mathcal{X}^*$ is separable and $\mathcal{M} : \mathcal{A} \to \mathcal{F}_c(\mathcal{X})$ is a $\mu$-continuous fuzzy valued measure of bounded variation, then $\mathcal{M}$ has Radon-Nikodým derivative contained in $\Lambda$.*

**Theorem 5** *Let $\mathcal{X}$ has the Radon-Nikodým property, $\mathcal{X}^*$ be separable and $\mathcal{F}$ sub-$\sigma$-algebra of $\mathcal{A}$. If $X \in \Lambda$ then there exists $E(X|\mathcal{F}) \in \Lambda(\Omega, \mathcal{F})$ such that*

$$\int_A X d\mu = \int_A E(X|\mathcal{F}) d\mu \text{ for all } A \in \mathcal{F}.$$

**Theorem 6** *Let $\mathcal{M} : \mathcal{A} \to \mathcal{F}_c(\mathcal{X})$ be a fuzzy measure. Then $\mathcal{M}$ has a unique Radon-Nikodým derivative contained in $\Lambda$ if and only if the following conditions are satisfied:*

1. *$\mathcal{M}$ is $\mu$-continuous,*
2. *$\mathcal{M}$ is of bounded variation,*
3. *given $A \subset \mathcal{A}, 0 < \mu(A) < \infty$, there exists a $B \subset A$*

*and a compact subset $C$ of $\mathcal{X}$ such that $\mu(B) > 0$ and $\frac{\mathcal{M}_0(B')}{\mu(B')} \subset C$ for all $B' \subset B$ with $\mu(B') > 0$*

$$(\text{where } \mathcal{M}_0(A) = \text{supp}\, \mathcal{M}(A) - \{x \in \mathcal{X} : \mathcal{M}(A)(x) > 0\}).$$

**Theorem 7** *If $\mathcal{M}^n : \mathcal{A} \to \mathcal{F}_c(\mathcal{X}), n = 1, 2$, are fuzzy measures of bounded variation which are $\mu$-continuous and such that for all $\mathcal{M}^1_\alpha(A) \subseteq \mathcal{M}^2_\alpha(A)$ for all $\alpha \in (0,1]$ and all $A \in \mathcal{A}$ then $(\frac{d\mathcal{M}^1_\alpha}{d\mu})(\omega) \subset (\frac{d\mathcal{M}^2_\alpha}{d\mu})(\omega)$ $\mu$ - a.e. where $d\mathcal{M}^n_\alpha$ and $d\mu$ are Radon-Nikodým derivative of $\mathcal{M}^n_\alpha$ and $\mu, \alpha \in (0,1]$.*

## References


[1] Bán J., Radon-Nikodým theorem and conditional expectation of fuzzy-valued measures and variables, Fuzzy Sets and Systems, 34 (1990), 383-392.

[2] Hiai F., Umegaki H., Integrals, conditional expectations and martingals of multivalued functions, J.Multivar.Anal. 7(1977), 149-182.

[3] Hiai F., Radon-Nikodým theorems for set valued measures, J.Multivar. Anal. 8, (1978), 96-118.

[4] Klement P., Puri M., Ralescu D., Limit theorems for fuzzy random variables, Proc. R.Soc. Land. A 407, (1986), 171-182.

[5] Kwakernaak H., Fuzzy random variables, Inform.Sci. 15(1978), 1-29.

[6] Lowen R., Convex fuzzy sets, Fuzzy Sets and Systems, 3, (1980), 291-310.

[7] Puri M., Ralescu D., Fuzzy random variables, J. Math. Anal. Appl. 114(1986), 409-422.

[8] Stojaković M., Fuzzy conditional expectation, Fuzzy Sets and Systems, (to appear)

[9] Stojaković M., Fuzzy random variable, martingales, (to appear)

[10] Zhang Wen-Xiu, Li Teng, Ma Ji-Feng, Li Ai-Jie, Set valued measure and fuzzy set-valued measure, Fuzzy Sets and Systems, 36 (1990) 181-188.