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COUNTER-EXAMPLES TO ZADEH'S POSSIBILITY THEORY

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ABSTRACT

In this short note we show that a number of conclusions unacceptable to our intuitions or commonsense knowledge can be drawn from Zadeh's possiliblity theory.

1. ZADEH'S DEFINITION OF POSSIBILITY

Suppose $U=\{u\}$ is the universe of discourse and F is a fuzzy set defined on U.

$$F=\{u,\ \mu_F(u)\} \tag{1}$$

Then the proposition $p=X$ is F, according to Zadeh [5], implies that F serves as a fuzzy restriction on the values which may be assumed to the varaible X, and a possibility distribution function $\pi(u)$ can be induced to associate with X such that

$$\pi(u)=\mu_F(u), \qquad u\in U \tag{2}$$

Further, for fuzzy set A defined on U

$$A=\{u,\ \mu_A(u)\} \tag{3}$$

Zadeh defined a possibility measure $\pi(A)$ in the following fashion

$$Poss\{X\ is\ A\}=\pi(A)$$
$$=\sup_{u\in U}(\mu_A(u)\wedge\pi(u)) \tag{4}$$

Here we should note two points.

① According to (4), the possibility that X is A is only determined by the point, say u_0, whose membership function achieves $\sup_{u\in U}(\mu_A(u)\wedge\pi(u))$ and take no account of other points in the universe of discourse. Thus we say Zadeh's possibility measure only takes account of 'single point' property. However we stress all the points in the universe of discouse make contributions to A and no specific point can alonely constitute A. Zadeh took the concept of fuzzy set as the basis for his possibility measure. So we would like to ask why a fuzzy set is defined by taking account of all the points in the universe of discourse, and on the contrast, whereas possibility measure is defined by taking account of only some specific point?

② Zadeh interpreted $\pi(u)$ as the possibility that $X=u$. In this way, Zadeh implicitly took the assumption

$$X=u_1 \ \Longrightarrow X\ is\ B=\{u,\ \mu_B(u)\} \tag{5}$$

where

$$\mu_B(u)=\begin{cases}1 & u=u_1\\ 0 & u\neq u_1\end{cases} \tag{6}$$

Then we wonder whether (5) always holds in reality ?

2. COUNTER-EXAMPLE I

Let us choose $A=F$. Then (4) indicates

$$X=F\Rightarrow Poss\{X\ is\ F\}=\sup_{u\in U}\mu_F(u) \tag{7}$$

Now we consider an extensively used example.

Let F signifty the concept 'old' defined on universe $U = [0, 200]$

$$\mu_F(u) = \begin{cases} 0 & ; \ 0 \leqslant u \leqslant 50 \\ \left[1 + (\frac{u-50}{5})^{-2} \right]^{-1} & ; \ 50 < u \leqslant 200 \end{cases} \quad (8)$$

We immediately conclude

$$X \text{ is } F \Rightarrow Poss\{X \text{ is } F\} < 1 \quad (9)$$

Then what is the possibility! Given X is F, we cannot assert the possibility that X is F is one, how can we interprete the physical essence of Zadeh's possibility measure? Then how can we use the possibility measure in proctice?

Of course, one may argue that the above example is not appropriate, either because the universe is not chosen properly or because the membership function is not chosen properly. And he or she may further argue that any fuzzy concept which coincides with reality is normalized. Unfortunately this is not the case. We can take software reliability problem as an example.

The number of logical paths in a software may be astronomical. A software can never be proved correct in a rigorous fashion [1]. We are not sure whether the logical paths are exhaustively activated. Even if a software never experiences failures under a given input environment, it is not sufficient to claim the software is correct. This is because there may exist defects in some logical paths which are never activated under the given input environment. However they may be activated under another input environment. So, when we propose 'software is absolutely reliable', fuzziness is attached to the term 'absolutely reliable'. Let T represent the time to the first software failure. Let AR denote the fuzzy term 'absolutely reliable'.

$$AR = \{T, \mu_{ar}(t)\} \qquad , T \in (0, \infty) \quad (10)$$

We may reasonably choose

$$\mu_{ar}(T) = c[1 - e^{-\lambda T}], \quad 0 < c < 1, \quad \lambda > 0 \quad (11)$$

In this way, according to Zadeh's possibility thoery,

$$\textit{Sofiware is absolutely reliable} \Rightarrow$$
$$Poss\{\textit{Sofiware is absolutely reliable}\}$$
$$c < 1 \quad (12)$$

Obviously, this is an unacceptable conclusion.

3. COUNTER–EXAMPLE II

We denote Poss $\{X \text{ is } A | X \text{ is } F\}$ as the possibility that X is A in the presence of X is F. Then from (4), we immediately arrive at

$$\begin{aligned} &Poss\ \{X \text{ is } A | X \text{ is } F\} \\ &= Poss\{X \in U | X \text{ is } A \text{ and } F\} \\ &= Poss\{X \text{ is } F | X \text{ is } A\} \end{aligned} \quad (13)$$

This is an unacceptable conclusion to our intuitions or commonsense knowledge. We can cite an example to explain this point. Let A represent 'old' and F 'very old'. Our commonsense knowledge tells us

$$\textit{John is very old} \Rightarrow \textit{John is old} \quad (14)$$

But

$$\textit{John is old} \nRightarrow \textit{John is very old} \quad (15)$$

Then our intuitions imply

$$\begin{aligned} &Poss\{\textit{John is old}|\textit{John is very old}\} > \\ &\quad Poss\{\textit{John is very old}|\textit{John is old}\} \end{aligned} \quad (16)$$

However (13) indicates

$$\begin{aligned} &Poss\{\textit{John is old}|\textit{John is very old}\} = \\ &\quad Poss\{\textit{John is very old}|\textit{John is old}\} \end{aligned} \quad (17)$$

So (13) has to be abandoned. Otherwise, we may ask how Zadeh's possibility theory can coincide with our intuitions and commonsense knowledge, or else, how it can provide useful help in practice?

4. INFORMATION CONVEYED BY PROPOSITIONS

Zadeh defined information conveyed by a proposition as possibility distribution induced by the proposition. Let $I(X \text{ is } F)$ be information conveyed by the proposition X is F. Zadeh claimed

$$F \subset G \Rightarrow I\ (X \text{ is } F) \geqslant I\ (X \text{ is } G) \quad (18)$$

If this is true, according to our intutitions, there should have

$$\begin{aligned} &Poss\{X \text{ is } G | X \text{ is } F\} \\ &\qquad \geqslant Poss\{X \text{ is } F | X \text{ is } G\} \end{aligned} \quad (19)$$

This contradicts (13). Then we would like to ask how to eliminate this self–contradiction in Zadeh's possibility theory? In fact, (18) may be false. Let's consider an example. Let t be the mean temperature in one year. Let E represent the concept ' adapt to plant rubber tree'. Then the corresponding membership function can be expressed as [4].

$$\mu_E(t)=\begin{cases}\dfrac{1}{1+0.0625(t-23)^2}, & 0\leqslant t<23(℃) \\ 1, & t\geqslant 23\end{cases} \quad (20)$$

Certainly, we can choose some integer n such that $\left[\dfrac{1}{1+0.0623\times 23^2}\right]^{1/2^n}>0.5$. Let G be the concept determined by

$$\mu_G(t)=[\mu_E(t)]^{1/2^n} \quad (21)$$

Further, we assume F represents the concept 'knowing nothing about the relations between the mean temperature in one year and the adaptability of planting rubber tree'. In this way, the corresponding membership function can be chosen as

$$\mu_F(t)=0.5 \quad \textit{for all } t\in[0,\infty) \quad (22)$$

Obviously $F\subset G$. However, according to our commonsense knowledge, G conveys more information than F, and thus (18) doesn't hold.

5. ESSENCE OF POSSIBILITY

Now, what is the sessence of possibility? How to interprete it in reality? Zadeh tried to reveal the essence of possibility by taking 'Hans eats eggs' as an example and pointed out that values assumed to possibilities may be different from those assumed to probabilities. But why this difference arises and what is the essence of possibility? He said nothing. In fact, it seems impossible to do so if we define possibility measure by taking account of only some specific point in the universe of discourse. We can see that all the counter–examples given in the above are attributed to this unappropriate account. Then how to overcome these counter–examples? To do so, we should insist that conclusions drawn from possibility meassure must coincide with our intuitions and commonsense knowledge such that it can provide a great usability in reality, and the corresponding possibility theory must exhibit mathematical elegance such that it suffers no self–contradiction. Then, given X is F, what is the possibility X is A ? Since both F and A are fuzzy sets, it is reasonable to take fuzzy sets as basis for defining the possibility. However, as we have pointed out, all the points in the universe of discourse make contributions to fuzzy sets. So, on basis of fuzzy sets, the desired possibility measure should take account of all the points in the universe of discourse. Given X is F, let $\eta(A)$ denote the possibility X is A. Let

$$F'=\{u,\ \mu_A(u)*\eta(A)\} \quad (23)$$

where $*$ is some operator we are currently unknown about. How to choose the value of $\eta(A)$? Our intuitions tell us such $\eta(A)$ should make F' closest to F. Of course, the meaning of 'closest' has to be defined in advance. Up to this point, we conclude that possibility characterizes particularity in the following sense. First, when we talk about the possibility X is A, we have to know some premise X is F. Second, given X is F, to choose the possibility X is A, we have to define the meaning of 'closest' in advance. So possibility measure makes sense only under some given premise and in some predefined sense. Otherwise it makes no sense. In fact, as pointed out by Cai [2, 3], possibility characterizes sample particularity, whereas probability stems from sample generality.

6. CONCLUSIONS

From the above discusions, we arrive at

(1). Zadeh's possibility theory should be reconsidered.

(2). Failures of Zadeh's possibility measure are attributed to the fact that it takes account of only some specific point in the universe of discourse.

(3). To define a proper possibility measure, all the points in the universe of discourse should be taken into account.

(4). Possibility makes sense only under given premise and in predefined sense. In this fashion, we say possibility characterizes particularity.

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