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INTERPOLATIVE REASONING FOR COMPUTATIONALLY EFFICIENT OPTIMAL FUZZY CONTROL

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Abstract. Fuzzy optimal control is considered. An optimal sequence of controls is sought best satisfying fuzzy constraints on the controls and fuzzy goals on the states (outputs), with a fuzzy system under control. Control over a fixed and specified, implicitly specified, fuzzy, and infinite termination time is discussed. For computational efficiency a small number of reference fuzzy staters and controls is to be assumed by which fuzzy controls and states are approximated. Optimal control policies reference fuzzy states are determined as a fuzzy relation used, via the compositional rule of inference, to derive an optimal control. Since this requires a large number of overlapping reference fuzzy controls and states implying a low computational efficiency, a small number of nonoverlapping reference fuzzy states and controls is assumed, and then interpolative reasoning is used to infer an optimal fuzzy control for a current fuzzy state.

Keywords: fuzzy control, fuzzy optimal control, fuzzy dynamic system, fuzzy inference, analogical reasoning, interpolative reasoning.

1. INTRODUCTION

As opposed to the usually assumed approach to fuzzy control, which boils down to the encoding of a set of (linguistic) control rules known from experience by the (human) operator, a different one, more consistent with the spirit of control, was advocated in Kacprzyk's (1983a) book: the temporal evolution of a fuzzy system under control, the fuzzy constraints and the fuzzy goals are assumed known, and an optimal sequence of controls is sought. The operator's knowledge concerns how the system behaves, and the fuzzy controls and states represent requirements on how control should proceed.

That book appeared in a "bad" time, long before the recent eruption of interest in fuzzy control. However, it may lead to a new "generation" of fuzzy control (*optimal fuzzy control*) (cf. Kacprzyk, 1992). Kacprzyk's (1983a) book concerns the following problem. $\mathcal{U} = \{C_1, \ldots, C_n\}$ is a set of *fuzzy controls* defined in U, $\mathcal{X} = \{S_1, \ldots, S_m\}$ is a set of *fuzzy states* (outputs) in X, the *fuzzy system under control* is governed by $X_{t+1} = F(X_t, U_t)$, where X_t, $X_{t+1} \in \mathcal{X}$ are fuzzy states at time (control stage) t and $t+1$, and $U_t \in \mathcal{U}$ is control at t, $t = 0, 1, \ldots, N-1$; N is the *termination time* (planning horizon). Next, $\mu_{\overline{C}^t}(U_t)$ is a *fuzzy constraint* on U_t, $\mu_{\overline{G}^{t+1}}(X_{t+1})$ is a *fuzzy goal* on X_{t+1}, and $\mu_D(. \mid .)$ is a *fuzzy decision*; $\overline{G}^{t+1}$ and $\overline{C}^t$ account for the fuzziness of X_{t+1} and U_t as, e.g., . $\mu_{\overline{G}^{t+1}}(X_t) = 1 - \mathrm{d}(X_{t+1}, G^{t+1})$ where d(.,.) is some distance; and similarly for $\overline{C}^t$.

We seek an *optimal sequence of fuzzy controls* $U_0^*, \ldots, U_{N-1}^*$ such that

$$\begin{aligned} \mu_D(U_0^*, \ldots, U_{N-1}^* \mid X_0) &= \\ &= \max_{U_0, \ldots, U_{N-1}} \mu_D(U_0, \ldots, U_{N-1} \mid X_0) = \\ &= \max_{U_0, \ldots, U_{N-1}} \wedge_{t=0}^{N-1} (\mu_{\overline{C}^t}(U_t) \wedge \mu_{\overline{G}^{t+1}}(X_{t+1})) \end{aligned} \quad (1)$$

Problem (1) leads to various problem classes which may be classified, e.g., due to: (1) the **termination time:** (a) fixed and specified in advance, (b) implicitly given (by entering a termination set of states), (c) fuzzy, and (d) infinite; and (2) the **system under control:** (a) deterministic, (b) stochastic, and (c) fuzzy. The cases of all termination types, and a fuzzy system under control will be discussed.

In all these problems the numerical efficiency requires finite, and relatively small numbers of fuzzy states and controls. Since in general these numbers are very high (theoretically infinite), we assume some reference fuzzy states and controls by which all fuzzy states and controls are approximated in the course of the algorithms. Then, we derive optimal policies relating optimal reference fuzzy controls to reference fuzzy states. These policies are represented by fuzzy relations which are in turn used, via the compositional rule of inference, to determine an optimal fuzzy control (not necessarily reference) for a current fuzzy state (not necessarily reference). For this procedure to give meaningful results, there should be a relatively large number of overlapping reference fuzzy states and controls. This is however harmful to the numerical efficiency of the optimal control algorithms used! We assume therefore a small number of nonoverlapping reference fuzzy states and controls, and then use an interpolative (analogical) reasoning scheme to infer an optimal fuzzy control for a current fuzzy state.

2. CONTROL WITH A FIXED AND SPECIFIED TERMINATION TIME

The problem is (1), and can be solved by dynamic programming (Baldwin and Pilsworth, 1982) which boils down to seeking $U_0^*, \ldots, U_{N-1}^*$ such that

$$\begin{aligned}\mu_D(U_0^*,\ldots,U_{N-1}^* \mid X_0) &= \\ &= \wedge_{t=1}^{N}(\max_{U_{t-1}}((\max(\mu_{U_{t-1}}(u_{t-1}) \wedge \mu_{C^{t-1}}(u_{t-1})) \wedge \\ &\wedge \max_{X_t}(\max_{x_t}(\mu_{X_t}(x_t \wedge \mu_{G^t}(x_t)))) \qquad (2)\end{aligned}$$

whose dynamic programming recurrence equations are

$$\begin{cases} \mu_{\bar{G}^N}(X_N) = \max_{x_N}(\mu_{X_N}(x_N \wedge \mu_{G^N}(x_N)) \\ \mu_{\bar{G}^{N-i}}(X_{N-i}) = \max_{U_{N-i}}(\max_{u_{N-i}}(\mu_{U_{N-i}}(u_{N-i})\wedge \\ \quad \wedge \mu_{C^{N-i}}(u_{N-i})) \wedge \mu_{\bar{G}^{N-i+1}}(X_{N-i+1})) \\ \mu_{X_{N-i+1}}(x_{N-i+1}) = \\ \quad = \max_{x_{N-i}}(\max_{u_{N-i}}(\mu_{U_{N-i}}(u_{N-i})\wedge \\ \quad \wedge \mu_{X_{N-i+1}}(x_{N-i+1} \mid x_{N-i}, u_{N-i}))\wedge \\ \quad \wedge \mu_{X_{N-i}}(x_{N-i})) \\ i = 1,\ldots,N \end{cases} \qquad (3)$$

This may be solved but the $\mu_{\bar{G}^t}(X_t)$'s are to be specified for all X_t's whose number may be huge (and similarly $\max_{U_{t-1}}$ is to proceed over a large set of U_{t-1}'s). A natural approach (Kacprzyk and Staniewski, 1982) is to use some prespecified *reference fuzzy states* and *reference fuzzy controls*, denoted $\overline{U}_t \in \overline{\mathcal{U}} = \{\overline{C}_1,\ldots,\overline{C}_l\} \subseteq \mathcal{U}$ and $\overline{X}_{t+1} \in \overline{\mathcal{X}} = \{\overline{S}_1,\ldots,\overline{S}_r\} \subseteq \mathcal{X}$, and to express all U_{t-1}'s and X_t's by their closest reference counterparts (fuzzy matching!). This is used to solve (3), and will be used throughout this paper.

The solution of (3), an *optimal control policy*, a_t^*, such that $\overline{U}_t^* = a_t^*(\overline{X}_t)$, $t = 0,1,\ldots,N-1$, is represented by "(IF $\overline{X}_t = \overline{S}_1$ THEN $\overline{U}_t = \overline{C}_{t1}$) ELSE ... ELSE IF($\overline{X}_t = \overline{S}_r$ THEN $\overline{U}_t = \overline{C}_{tr}$)", $\overline{C}_{t1},\ldots,\overline{C}_{tr} \in \overline{\mathcal{U}}$ equated with a fuzzy relation R in $X \times U$. Thus, for a current X_t the U_t^* sought is determined by the *compositional rule of inference* $U_t^* = X_t \circ R$.

Unfortunately, this dynamic programming scheme is not efficient. Moreover, for the compositional rule of inference to work properly there should be a high number of overlapping fuzzy controls and states. This is however harmful to the computational efficiency of dynamic programming! This contradiction will be resolved by assuming a small number of nonoverlapping fuzzy states and controls, and using then interpolative reasoning as considered in Section 6.

Simpler and more efficient than dynamic programming sketched above is an earlier branch-and-bound approach by Kacprzyk (1978a, 1979): we seek $\overline{U}_0^*, \ldots, \overline{U}_{N-1}^*$ such that

$$\begin{aligned}\mu_D(\overline{U}_0^*,\ldots,\overline{U}_{N-1}^* \mid \overline{X}_0) &= \max_{\overline{U}_0,\ldots,\overline{U}_{N-1}}(\mu_{\overline{C}^0}(\overline{U}_0)\wedge \\ &\wedge \mu_{\overline{G}^1}(\overline{X}_1) \wedge \ldots \wedge \mu_{\overline{C}^{N-1}}(\overline{U}_{N-1}) \wedge \mu_{\overline{G}^N}(\overline{X}_N)) \quad (4)\end{aligned}$$

The solution of (4) is based on the following property of "$\wedge$" (and of many other t- norms too): if $\mu_D(\overline{U}_0, \ldots, \overline{U}_k \mid X_0) = \wedge_{t=0}^{k-1}(\mu_{\overline{C}^t}(\overline{U}_t)\wedge\mu_{\overline{G}^{t+1}}(\overline{X}_{t+1})$, then $N-1>k>l \Rightarrow \mu_D(U_0, \ldots, U_{N-1} \mid X_0) \leq \mu_D(U_0, \ldots, U_k \mid X_0) \leq \mu_D(U_0, \ldots, U_l \mid X_0)$. That is, by "adding" next controls we cannot increase the value of $\mu_D(. \mid .)$. The branching is via the $\overline{U}_t$'s and the bounding is via the $\mu_D(\overline{U}_0, \ldots, U_t \mid \overline{X}_0)$'s.

An efficient branching requires a small number of reference fuzzy controls, $\overline{C}_1, \ldots, \overline{C}_w$. As a solution we obtain an optimal policy $\overline{U}_t^* = a^*(\overline{X}_t)$ which is determined as a fuzzy relation, and for a current (not necesarily reference) fuzzy state the optimal control is determined via the compositional rule of inference. And again, there should be sufficiently many, overlapping reference fuzzy states and controls to obtain meaningful results. For numerical efficiency we should however have as few as possible, nonoverlapping reference fuzzy states and controls. Then, interpolative reasoning can be used.

3. CONTROL WITH AN IMPLICITLY SPECIFIED TERMINATION TIME

The termination time N is now when the system enters for the first time a terminating set of states, $\mathcal{X}^T \subset \mathcal{X}$. Suppose that $\overline{\mathcal{X}} = \{\overline{S}_1, \ldots, \overline{S}_p, \overline{S}_{p+1}, \ldots, \overline{S}_r\}$ and $\overline{\mathcal{X}}^T = \{\overline{S}_{p+1}, \ldots, \overline{S}_r\}$.

We seek $\overline{U}_0^*, \ldots, \overline{U}_{K-1}^*$ such that

$$\begin{aligned}\mu_D(\overline{U}_0^*,\ldots,\overline{U}_{K-1}^* \mid \overline{X}_0) &= \\ &= \max_{\overline{U}_0,\ldots,\overline{U}_{K-1}}(\mu_{\overline{C}}(\overline{U}_0 \mid \overline{X}_0) \wedge \ldots \\ &\ldots \wedge \mu_{\overline{C}}(\overline{U}_{K-1} \mid \overline{X}_{K-1}) \wedge \mu_{\overline{G}}(\overline{X}_K)) \qquad (5)\end{aligned}$$

where $\overline{X}_K \in \mathcal{X}^T$ and $\overline{X}_{K-1} \in \mathcal{X} - \mathcal{X}^T$. Using the approximation by reference fuzzy controls and states, (5) may be solved using: an iterative approach, a graph-theoretic approach, and a branch-and-bound approach (cf. Kacprzyk, 1983a). The first is related to the case of an infinite termination time (cf. Section 5), the second is not operational, and the last is simple and efficient – analogous to that discussed in Section 3.

This model may be very useful, in particular for optimizing the first part of the trajectory until a stable operation when the termination time is evidently unknown in advance.

4. CONTROL WITH A FUZZY TERMINATION TIME

The idea of a fuzzy termination time (*a couple of, some, about ten, ...* control stages) appeared in Fung and Fu (1977) and Kacprzyk (1977, 1978a, c). It is given as a fuzzy set, W, in $V = \{1,\ldots,K,K+1,\ldots,N\}$; $\mu_W(t) \in [0,1]$ represents how "good" t is as the termination time. The process should terminate at some $M \in \text{supp}V = \{t \in T : \mu_W(t) > 0\}$; $M \in \{K,K+1,\ldots,N\}$ is assumed.

We seek an optimal termination time M^* and $\overline{U}_0^*, \ldots, \overline{U}_{M^*}^*$ such that

$$\begin{aligned}\mu_D(\overline{U}_0^*,\ldots,\overline{U}_{M^*}^* \mid X_0) &= \max_{M,\overline{U}_0,\ldots,\overline{U}_{M-1}}(\mu_{\overline{C}^0}(\overline{U}_0) \wedge \ldots \\ &\ldots \wedge \mu_{\overline{C}^{M-1}}(\overline{U}_{M-1}) \wedge (\mu_W(M)\, \mu_{\overline{G}^M}(\overline{X}_M))) \quad (6)\end{aligned}$$

For solving (6), Kacprzyk's (1977, 1978a, c) dynamic programming algorithm similar to (2), and Kacprzyk's (1978b, 1979) branch-and-bound scheme (cf. Section 3) may be used.

In the former case, we devise two sets of recurrence equations:

$$\begin{cases} \overline{\mu}_{\tilde{G}^M}(\overline{X}_M, M) = \\ \quad = \max_{x_M}(\mu_{X_M}(x_M) \wedge \mu_W(M)\, \mu_{\overline{G}^M}(\overline{X}_M)) \\ \overline{\mu}_{\tilde{G}^{M-i}}(\overline{X}_{M-i}, M) = \\ \quad = \max_{U_{M-i}}(\max_{u_{M-i}}(\mu_{\overline{U}_{M-i}}(u_{M-i}) \wedge \\ \quad \wedge\, \mu_{\overline{C}^{M-i}}(\overline{U}_{M-i})) \wedge \overline{\mu}_{\tilde{G}^{M-i+1}}(\overline{X}_{M-i+1}, M)) \\ \mu_{\overline{X}_{M-i+1}}(x_{M-i+1}) = \\ \quad = \max_{x_{M-i}}(\max_{u_{M-i}}(\mu_{U_{M-i}}(u_{M-i}) \wedge \\ \quad \wedge\, \mu_{\overline{X}_{M-i+1}}(x_{M-i+1} \mid x_{N-i}, u_{N-i})) \wedge \\ \quad \wedge\, \mu_{\overline{X}_{M-i}}(x_{M-i})) \\ i = 1, \ldots, M-K+1; M = K, \ldots, N \end{cases} \quad (7)$$

$$\mu_{\overline{G}^{K-1}}(\overline{X}_{K-1}) = \max_M \overline{\mu}_{\tilde{G}^{K-1}}(\overline{X}_{K-1}, M) \quad (8)$$

$$\begin{cases} \mu_{\overline{G}^{K-i-1}}(\overline{X}_{K-i-1}) = \\ \quad = \max_{\overline{U}_{K-i-1}}(\max_{u_{K-i-1}}(\mu_{\overline{U}_{K-i-1}}(u_{K-i-1}) \wedge \\ \quad \wedge\, \mu_{\overline{C}^{K-i-1}}(\overline{U}_{K-i-1})) \wedge \mu_{\overline{G}^{K-i}}(\overline{X}_{K-i})) \\ \mu_{\overline{X}_{K-i}}(x_{K-i}) = \\ \quad = \max_{x_{K-i-1}}(\max_{u_{K-i-1}}(\mu_{\overline{U}_{K-i-1}}(u_{K-i-1}) \wedge \\ \quad \wedge\, \mu_{\overline{X}_{K-i}}(x_{K-i} \mid x_{K-i-1}, u_{K-i-1})) \wedge \\ \quad \wedge\, \mu_{\overline{X}_{K-i-1}}(x_{K-i-1})) \\ i = 1, \ldots, K-1 \end{cases} \quad (9)$$

We obtain optimal policies $\overline{U}_t^* = a_t^*(\overline{X}_t)$, and all the previous remarks on the number of reference fuzzy states and controls and on interpolative reasoning are also valid here.

5. CONTROL WITH AN INFINITE TERMINATION TIME

When the process is to proceed over a long time and is low-varying, with the goal just to maintain some (stable) conditions, it may be expedient to assume an infinite termination time and use some specific apparatus proposed first by Kacprzyk and Staniewski (1982, 1983) (cf. also Kacprzyk, 1983a).

We seek an *optimal stationary policy* a_∞^* such that

$$\mu_D(a_\infty^* \mid X_0) = \max_{a_\infty} \mu_D(a_\infty \mid X_0) = \\ = \lim_{N \to \infty} \wedge_{t=0}^{N} b^t(\mu_{\overline{C}}(a \mid X_t) \wedge \mu_{\overline{G}}(X_{t+1})) \quad (10)$$

where $b > 1$ is a *discount factor* expressing a higher importance of earlier control stages.

A policy iteration algorithm was given by Kacprzyk and Staniewski (1982, 1983) to solve problem (10) which, via the approximation by reference fuzzy states and controls, was transformed into one with an auxiliary finite state deterministic system. Thus, if $A(.)$ means this approximation, the auxiliary deterministic system representing the fuzzy system is given by $\overline{X}_{t+1} = A(F(\overline{X}_t, \overline{U}_t)), t = 0, 1, \ldots$, and, e.g., $\mu_{\overline{C}}(\overline{U}_t \mid \overline{X}_t) = e((\overline{C}(\overline{X}_t), \overline{U}_t)$ and $\mu_{\overline{G}}(\overline{X}_t) = e(\overline{G}, \overline{X}_t)$, where $e(.,.)$ is a degree of equality of two fuzzy sets.

We seek an optimal stationary policy a_∞^* such that

$$\mu_D(a_\infty^*) = \max_{a_\infty} \lim_{N \to \infty} \wedge_{t=0}^{N} b^t(\mu_{\overline{C}}(a \mid \overline{X}_t) \wedge \mu_{\overline{G}}(\overline{X}_{t+1})) \quad (11)$$

An a_∞^* solving (11) is determined by a policy iteration type algorithm (Kacprzyk and Staniewski, 1982, 1983) whose essence is:

Step 1. Choose an arbitrary $a_\infty = (a, a, \ldots)$.
Step 2. Solve in $\mu_D(a_\infty \mid \overline{S}_i)$, $i = 1, \ldots, r$:

$$\mu_D(a_\infty \mid \overline{S}_i) = \\ = \mu_{\overline{C}}(a(\overline{S}_i) \mid \overline{S}_i) \wedge \mu_{\overline{G}}(A(F(\overline{S}_i, a(\overline{S}_i)))) \wedge \\ \wedge b\mu_D(a_\infty \mid A(F(\overline{S}_i, a(\overline{S}_i)))) \quad (12)$$

Step 3. Improve a_∞, i.e. find a z^* maximizing $\mu_{\overline{C}}(z(\overline{S}_i) \mid \overline{S}_i) \wedge \mu_{\overline{G}}(A(F(\overline{S}_i, z(\overline{S}_i)))) \wedge b\mu_D(a_\infty \mid A(F(\overline{S}_i, a(\overline{S}_i))))$.
Step 4. If z^* found in Step 3 is the same as the previous, it is an optimal stationary policy sought. Otherwise, assume $a_\infty = z^*$ and return to Step 2.

We obtain (in a finite number of steps!) an a_∞^* for the reference fuzzy states only. Needless to say that the number of them (and of reference fuzzy controls) has a decisive impact on the efficiency of the algorithm. Thus, as in previous sections, we assume a small number of reference fuzzy states and fuzzy controls, solve the problem, and then use interpolative resoning.

6. INTERPOLATIVE REASONING IN THE DERIVATION OF OPTIMAL CONTROLS

In all the above cases there is a conflict between a large number of reference fuzzy controls and states required for obtaining meaningful results via the compositional rule of inference, and a small number of them required for the numerical efficiency.

Since for real problems the efficiency may be decisive, we may be forced to assume a small number of nonoverlapping reference fuzzy controls and states, and the situation will be: we obtain an optimal policy a_t^* stating "IF $\overline{X}_t = \overline{S}_1$ THEN $\overline{U}_t^* = \overline{C}_{t1}$ ELSE ... ELSE IF $\overline{X}_t = \overline{S}_i$ THEN $\overline{U}_t^* = \overline{C}_{ti}$ ELSE IF $\overline{X}_t = \overline{S}_{i+1}$ THEN $\overline{U}_t^* = \overline{C}_{t(i+1)}$ ELSE ... ELSE IF $\overline{X}_t = \overline{S}_r$ THEN $\overline{U}_t^* = \overline{C}_{tr}$". The problem is the implementation of a_t^*. Suppose that we wish to determine U_t^* for a current X_t, not a reference one. Let X_t be a fuzzy number between the two reference fuzzy states $\overline{S}_i$ and $\overline{S}_{i+1}$. We seek therefore an U_t^* corresponding to X_t via this a_t^*. Notice that since X_t is not a reference one, U_t^* will not be in general a reference one either.

The determination of U_t^* is meant here, assuming a representation by triangular fuzzy numbers, as the determination of the mean value and width. It is reasonable to require U_t^* to be similar (close) to one of these optimal $\overline{C}_{ti}$ and $\overline{C}_{t(i+1)}$ corresponding to $\overline{S}_i$ and $\overline{S}_{i+1}$.

The idea of our approach is as follows. The first problem is to determine the mean value of the fuzzy optimal control sought. We apply here Kóczy and Hirota's (1992) approach whose essence may be expressed as

$$d(\overline{S}_i, X_t)/d(X_t, \overline{S}_{i+1}) = d(\overline{C}_i, U_t^*)/d(U_t^*, \overline{C}_{t(i+1)}) \quad (13)$$

where $d(.,.)$ is a distance between two fuzzy sets. The sense of (13) is that the relative position of U_t^* with respect to its closest reference counterparts should be the same as that concerning X_t and its reference counterparts.

The second problem is to determine the width of U_t^*. The reasoning is that the lower the number of reference fuzzy states and controls, the less precise is the available information. Hence, the fuzzier (of a larger width) U_t^* should be. For instance, we can use a formula

$$\overline{w}(U_t^*) = \frac{1}{5}[\overline{w}(\overline{S}_i) + \overline{w}(\overline{S}_{i+1}) + \\ + \overline{w}(\overline{X}_t) + \overline{w}(\overline{C}_{ti}) + \overline{w}(\overline{C}_{t(i+1)})] \qquad (14)$$

where $\overline{w}(.)$ is a relative width (related to the universe of discourse of the fuzzy states and controls, and the simplest arithmetics mean (14) can be replaced by another formula expressing the above rationale.

Moreover, in addition to the right-hand-side terms of 14) which express the fuzziness of the reference fuzzy controls and states involved, it may often be expedient to include some term(s) accounting for the "overall fuzziness" of the control problem considered. This requires a new conceptual approach which will be presented in a sunsequent paper.

7. CONCLUDING REMARKS

Models of optimal fuzzy control were discussed. To efficiently solve these control problems, a small number of reference fuzzy states and controls was assumed, and optimal policies were derived. To make it possible to use these policies not necessarily for the reference fuzzy states and controls, an interpolative reasoning scheme was proposed which determines the position of a (non-reference) fuzzy optimal control sought and its width (fuzziness).

REFERENCES

Baldwin J.F. and B.W. Pilsworth (1982) Dynamic programming for fuzzy systems with fuzzy environment. *J. of Math. Anal. and Appls.* **85**, 1–23.

Fung L.W. and K.S. Fu (1977) Characterization of a class of fuzzy optimal control problems. In M.M. Gupta, G.N. Saridis and B.R. Gaines (Eds.): *Fuzzy Automata and Decision Processes*, New York: North-Holland, pp. 209–219.

Kacprzyk J. (1977) Control of a non-fuzzy system in a fuzzy envirnment with fuzzy termination time. *Systems Science* **3**, 320–331.

Kacprzyk J. (1978a) Control of a stochastic system in a fuzzy environment with fuzzy termination time. *Systems Science* **4**, 291–300.

Kacprzyk J. (1978b) A branch-and-bound algorithm for the multistage control of a nonfuzzy system in a fuzzy environment. *Control and Cybernetics* **7**, 51–64.

Kacprzyk J. (1978c) Decision-making in a fuzzy environment with fuzzy termination time. *Fuzzy Sets and Systems* **1**, 169–179.

Kacprzyk J. (1979) A branch-and-bound algorithm for the multistage control of a fuzzy system in a fuzzy environment. *Kybernetes* **8**, 139–147.

Kacprzyk J. (1983a) *Multistage Decision-Making under Fuzziness.* Cologne: Verlag TÜV Rheinland.

Kacprzyk J. (1983b) A generalization of fuzzy multistage decision making and control via linguistic quantifiers. *Int. J. of Control* **38**, 1249–1270.

Kacprzyk J. (1992) Fuzzy optimal control revisited: toward a new generation of fuzzy control. *Proc. of the 2nd Int. Conference on Fuzzy Logic and Neural Networks (Iizuka '92)*, Vol. 2, pp. 429–432.

Kacprzyk J. and C. Iwański (1987) Generalization of discounted multistage decision making and control through fuzzy linguistic quantifiers: an attempt to introduce commonsense knowledge. *Int. J. of Control* **45**, 1909–1930.

Kacprzyk J. and P. Staniewski (1982) Long term inventory policy making through fuzzy decision making models. *Fuzzy Sets and Systems* **8**, 117–132.

Kacprzyk J. and P. Staniewski (1983) Control of a deterministic system in a fuzzy environment over infinite planning horizon. *Fuzzy Sets and Systems* **10**, 291-298.

Kóczy L. and K. Hirota (1992) Analogical fuzzy reasoning and gradual inference rules. *Proc. of the 2nd Int. Conference on Fuzzy Logic and Neural Networks (Iizuka '92)*, Vol. 1, pp. 329–332.