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Numerical Solution of Inverse Problem of Fuzzy Modeling with Pseudo First Order Approximation

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Abstract

Numerical solution of inverse problem of Takagi-Sugeno fuzzy model is proposed. The method is located on the application of numerical optimization to the fuzzy model. Steepest descent method is used for the numerical optimization. We use the linear approximation of fuzzy model, called pseudo first order approximation, by fixing the membership value on the neighborhood of the corresponding input. It is introduced in order to reduce the difficulty of optimization process. The efficiency of this method is shown by a numerical experiment.
Key Words : *Fuzzy Model, Inverse Problem, Numerical Optimization*

1 Introduction

In the recent research of applying fuzzy systems to various fields, we can see the following two kinds of development in these systems. The first is implementation of experts' skill to a machine system, and the second is the self learning system that obtain the knowledge by machine itself. In the latter case, the knowledge is described by internal parameters of the system and they are estimated by some kinds of learning method.

It is also known that the inverse problem is to estimate input variables under the given system and output variables. This can be interpreted as a learning system by replacing the estimating parameters from internal parameters of the learning system to input variables of the inverse problem. So it is possible to apply the estimating method of the learning system to the determination of the input variables. In the application of fuzzy relation, inverse problem is used for the diagnosis of machine failure or medical symptoms [4, 7]. The solving methods of inverse problem of fuzzy relation have already been reported [1, 3].

The Takagi-Sugeno fuzzy inference model [6] is also the fuzzy system that has the inference rules whose conclusion part consists of a linear equation of input variables. There exist many researches of estimating the inference parameters, such as membership functions of condition part or coefficients of linear equation e.g. [5]. Since we treat the input variables to be estimated in this research, this gives a numerical solution of the inverse problem of the fuzzy model. We apply the numerical optimization method for this problem to obtain a numerical solution of the fuzzy model.

It is difficult to apply the numerical optimization process to the fuzzy model directly because of its nonlinearity. We introduce a kind of linear approximation of the fuzzy model by fixing the membership value on the neighborhood of the corresponding input. We call this linear approximation *a pseudo first order approximation* . By introducing this approximation, it becomes easy to calculate the partial difference, which is necessary to obtain for the numerical optimization. The efficiency of the proposed method has been confirmed through the numerical example.

2 Fuzzy modeling and its pseudo first order approximation

A fuzzy modeling has been proposed by Takagi and Sugeno [6]. The summary of the fuzzy modeling and the definition of the pseudo first order approximation of the model are shown in the followings. Now let's consider the system with N inputs and a single output. Inputs are denoted by a vector $\mathbf{x} = [x_1, x_2, \ldots, x_N]^t$. The output y of the fuzzy model is

$$y = \sum_{j=1}^{R} w_j(\mathbf{x}) \cdot f_j(\mathbf{x}) / \sum_{j=1}^{R} w_j(\mathbf{x}), \tag{1}$$

where the index j corresponds to the j-th rule. In this formula, $f_j(\cdot)$ is a linear function of the input variables, i.e.

$$f_j(\mathbf{x}) = a_{j0} + \sum_{i=1}^{N} a_{ji}x_i, \tag{2}$$

and $w_j(\cdot)$ is the membership value of each rule, such that

$$w_j(\mathbf{x}) = \prod_{i=1}^{N} \mu_i^{\phi(j,i)}(x_i), \tag{3}$$

where μ is the membership function of the i-th input variable of the j-th rule, and $\phi(j,i)$ is the mapping of the index which corresponds to the choice of membership label of the condition part. The membership function $\mu_i^j(\cdot)$ is described by two kinds parameters $l_{..}$'s and $r_{..}$'s ($l_{..} < r_{..}$),

$$\mu_i^j(x) = \begin{cases} 0.0 & x \leq r_{j-1,i} \\ & \text{or } x \geq l_{j+1,i} \\ 1.0 & l_{ji} \leq x \leq r_{ji} \\ \frac{(x-r_{j-1,i})}{(l_{ji}-r_{j-1,i})} & r_{j-1,i} < x < l_{ji} \\ \frac{(x-l_{j+1,i})}{(r_{ji}-l_{j+1,i})} & r_{ji} < x < l_{j+1,i} \end{cases}. \tag{4}$$

The rule construction is done by determining the l,r's of membership functions, the mapping ϕ, and the coefficients of the linear function of the conclusion part. Here, we assume that these parameters are given by human experts or some kind of automatic estimation algorithms.

Now we define the pseudo approximation of fuzzy model on neighborhood of $\hat{\mathbf{x}}$. When we consider the linear approximation of the fuzzy model, it is difficult to write down the approximation formula directly because the output of the model consists of a fractional function of input variables. We assume that the membership function has the constant value on the neighborhood of $\hat{\mathbf{x}}$. From this assumption, we can obtain the pseudo approximated fuzzy model of the first order

$$y \approx \sum_{j=1}^{R} w_j(\hat{\mathbf{x}})f_j(\mathbf{x}) / \sum_{j=1}^{R} w_j(\hat{\mathbf{x}}). \tag{5}$$

By using this approximation, we can easily obtain the pseudo partial difference of $\mathbf{x}$ as

$$\frac{\partial y}{\partial x_i} \approx \sum_{j=1}^{R} w_j(\hat{\mathbf{x}})a_{ji} / \sum_{j=1}^{R} w_j(\hat{\mathbf{x}}). \tag{6}$$

3 Solving algorithm of inverse problem

At first, we define the solution of inverse problem of fuzzy model. The solution is defined by minimizing the square error

$$e(\mathbf{x}) = \frac{1}{2}[y(\mathbf{x}) - y^*]^2, \tag{7}$$

where y^* is the desired output of fuzzy model.
This kind of problem can be formulated to the numerical optimization problem,

$$\begin{array}{l} \text{minimize}\, e(\mathbf{x}) \\ \text{subject to}\, h_i(x_i) \leq 0\, \text{for all i.} \end{array} \tag{8}$$

In this description, the inequality constraint $h_i(\cdot)$ denotes the feasible region of $\mathbf{x}$. When the x_i takes the values in the range between b_i and c_i, i.e. $b_i \leq x_i \leq c_i$, we can describe it

$$h_i(x_i) = (x_i - b_i)(x_i - c_i), b_i < c_i. \tag{9}$$

In order to solve this optimization problem, Augmented Lagrangian function [2] is introduced as follows

$$\begin{aligned} & L_c(\mathbf{x}, \mathbf{z}, \lambda) \\ &= e(\mathbf{x}) + \sum_{i=1}^{N} \lambda_i \left\{h_i(x_i) + (z_i)^2\right\} \\ &\quad + \frac{1}{2} r \sum_{i=1}^{N} \left\{h_i(x_i) + (z_i)^2\right\}^2, \end{aligned} \tag{10}$$

where λ is a vector of Lagrangian multiplier λ_i, and $\mathbf{z} = [z_1, z_2, \ldots, z_N]^t$ is used in order to treat inequality constraint. The r is a penalty parameter determined empirically.
By solving $\mathbf{z}$ to minimize L_c, we obtain

$$(z_i)^2 = \begin{cases} -\frac{\lambda_i}{r} - h_i(x_i) & \lambda_i + r \cdot h_i(x_i) < 0 \\ 0 & otherwise \end{cases}, \tag{11}$$

and the Lagrangian function

$$\begin{aligned} & L(\mathbf{x}, \lambda) = e(\mathbf{x}) \\ & + \frac{1}{2r} \sum_{i=1}^{N} \left[\max\{0, \lambda_i + r \cdot h_i(x_i)\}^2 - (\lambda_i)^2\right]. \end{aligned} \tag{12}$$

In the algorithm to obtain the numerical solution of this problem, we use the following procedure for the modification of $\mathbf{x}$ and λ,

$$\mathbf{x}_{t+1} = \mathbf{x}_t + \epsilon \Delta \mathbf{x}_t, \tag{13}$$

$$\lambda_{t+1} = \lambda_t + \Delta \lambda_t, \tag{14}$$

where $t = 0, 1, 2, \ldots, T$. Hence in the case of $e(\mathbf{x}) < \theta$, the iteration is terminated since we have already obtained the solution in enough precision.
In these formulae, $\Delta \mathbf{x}_t$, which is called a search direction, and $\Delta \lambda_t$ are decided by the following calculations. The step size ϵ , which takes a value as $0 < \epsilon \leq 1$, is determined empirically. The initial value $\mathbf{x}_0$ is to be chosen by the appropriate way in order to avoid the local minimum solution.

For the search direction, we can apply the steepest descent method as the numerical optimization, such that

$$\Delta \mathbf{x}_t = -\nabla_{\mathbf{x}} L(\mathbf{x}, \lambda)|_{\mathbf{x}=\mathbf{x}_t, \lambda=\lambda_t}. \tag{15}$$

where

$$\nabla_{\mathbf{x}} L = \left[\frac{\partial L}{x_1}, \frac{\partial L}{x_2}, \ldots, \frac{\partial L}{x_N}\right]^t. \tag{16}$$

The more precise description of each partial difference of Lagrangian function is given as,

$$\frac{\partial L}{\partial x_i} = (y - y^*)\frac{\partial y}{\partial x_i} + \begin{cases} \frac{\partial h_i}{\partial x_i}(\lambda_i + r \cdot h_i(x_i)) & h_i(x_i) \leq -\frac{\lambda_i}{r} \\ 0 & otherwise \end{cases}. \tag{17}$$

In this formula, partial differences of y by x_i appear. In order to calculate this, we use the approximated fuzzy model on the neighborhood of x_t and its partial differences.

The modification of Lagrangian multiplier is done by according to the following procedure

$$\Delta\lambda_i(t) = \begin{cases} r \cdot h_i(x_i(t)) & \lambda_i(t) + r \cdot h_i(x_i(t)) \geq 0 \\ 0 & otherwise \end{cases}. \tag{18}$$

4 Extension for multiple-output

A single-output case of the fuzzy model was treated in the previous section. Now, we can easily extend the algorithm of inverse problem to the multiple-output fuzzy model. The multiple-output fuzzy model is described as follows,

$$\mathbf{y} = [y_1, y_2, \cdots, y_M]^t, \tag{19}$$

$$y_i = \sum_{j=1}^{R_i} w_{ij}(\mathbf{x}) \cdot f_{ij}(\mathbf{x}) / \sum_{j=1}^{R_i} w_{ij}(\mathbf{x}), \tag{20}$$

where w_{ij} and f_{ij} are membership functions and linear equations of the conclusion part for multiple-output extension, respectively. In this notation, i shows the index of the corresponding output y_i. Here, a pseudo approximation of fuzzy model is given by the same notaion as the formula of the fuzzy model.

An objective function for the numerical optimization problem is given as

$$E(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{M}[y_i(\mathbf{x}) - y_i^*]^2. \tag{21}$$

By using this function, the numerical optimization is done in the same manner as the case of the single-output mentioned above.

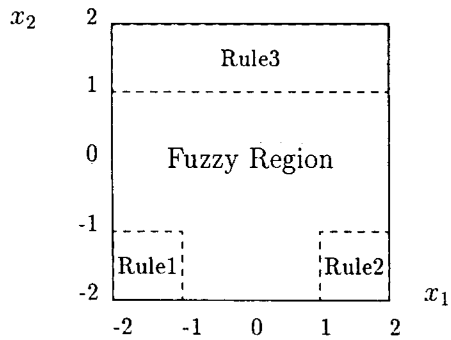

Figure 1: Fuzzy partition used in numerical experiment

Table 1: Membership functions and constraint parameters

Variable		x1		x2	
membership	label	μ_1^1	μ_1^2	μ_2^1	μ_2^2
function	l	-2	1	-2	1
	r	-1	2	-1	2
range	b	-2		-2	
	c	2		2	

5 Examples

In order to confirm the efficiency of the proposed method, numerical examples are shown. There are two inputs in this example. Each input has a feasible range for its value shown in Table 1. Membership functions are assigned to each variable as shown in Table 1. The rules used in this example are listed in Table 2. In the item *premise mappings* of this table , a hyphen indicates that there exists no such mapping. In this case we put the value μ as 1.0. We obtain the fuzzy partition from these rules as shown in Figure 1.

We also need the initial value of input variables for the estimation of input variables. Initial values used here are shown in Table 3.

The specified parameters of the numerical optimization used in this experiment, have been determined empirically, and are shown in Table 4.

Table 2: Fuzzy Rules

Rule	Premise mappings		Conclusion coefficients		
	$\phi(\cdot,1)$	$\phi(\cdot,2)$	a_0	a_1	a_2
1	1	1	0.0	1.0	1.0
2	2	1	0.0	1.0	0.0
3	-	2	0.0	0.0	-1.0

Table 3: Initial values for the input variables

No.	1	2	3	4	5
$x_1(0)$	-1.5	-1.5	1.5	1.5	0.0
$x_2(0)$	-1.5	1.5	-1.5	1.5	0.0

Table 4: Specified Parameters of Numerical Optimization

	Parameter	Value
ϵ	Step size of input values	0.1
r	Penalty parameter	0.1
T	Maximum cycle of modification	1000
θ	Threshold of $e()$ for termination	0.00001

The experiment is done by the following steps.
Step-1: For the both two inputs, which take values between -1.5 and 1.5 with interval 0.1, calculate the output of the fuzzy model. We will obtain 961 outputs by this calculation. These are used for the desired output on the subsequence step.
Step-2: For all desired outputs calculated in Step-1, apply the proposed algorithm by using each five initial values given in Table 3. Then we will obtain the estimated ($4805 = 961 \times 5$) inputs.
Step-3: For each five solutions which correspond to the five initial values, pick up the optimal solution which serves the minimal value of $e(\cdot)$ for each five solutions. Then we obtain the result of experiment as the optimal solutions which correspond to the each desired outputs (there exist 961 solutions).

According to the above mentioned procedure, we obtain the final result of experiment. In order to evaluate the obtained solutions, we calculated the square error between the desired output and the output provided by the solution. For all solutions, we can obtain the value $e(\cdot)$ smaller than the threshold θ. This means the proposed method gave the solution in the inverse problem for all outputs in this experiment.

For detailed information on this experiment, i.e. how many initial values are active for the optimal solution, the summary is shown in Table 5. By referring the Table 5, we can see that the most of data can be solved by starting any initial value, and about five percent of data are solved by using only two initial values.

Table 5: Number of Data of each five initial values. "Number of success" means which can find out the global minimum($e(\mathbf{x}) \leq \theta$).

Number of Success	5	4	3	2	1
Number of Data	917	0	0	44	0

6 Conclusion

An algorithm to obtain the numerical solution of an inverse problem of fuzzy model is proposed. The solution of the inverse problem is defined in a sense of a minimum solution of the square error between the desired output and the actual output value. A pseudo first order approximation is defined by fixing the membership value on the neighborhood of the corresponding input. By using this approximation, it becomes easy to formulate the inverse problem for the numerical optimization problem. The efficiency of the proposed method is shown through a numerical experiment. The result of experiment is that all of the desired output can be obtained with enough minimal error by using the estimated solution.

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