Fifth IFSA World Congress (1993), 1159-1162

The ξ-Quality Defuzzification Method

Hans Hellendoorn **Christoph Thomas**

Siemens AG, ZFE ST SN 42

Abstract

We describe six important defuzzification methods and their respective merits and shortcomings, dependent on the rules, domains, *etc.* Furthermore, we present an alternative approach, the so called ξ-Quality defuzzification method, for the case that the output fuzzy sets have different shape or are asymmetric.

KEYWORDS: Fuzzy Set Theory, Fuzzy Control, Defuzzification.

1 Several Defuzzification Methods

As the topic 'Fuzzy-Control' [3] is a field of research which is in discussion since only about ten years, the use of terms describing the various basics of a fuzzy system is unfortunately still ambiguous. In order to facilitate the understanding of the following investigations, a definition of the defuzzification methods under concern in this paper will be given. The resulting behavior of fuzzy controllers using any of these defuzzification methods will be dicussed in the following chapters. Our investigation covers six defuzzification methods: Center-of-Area/Gravity defuzzification, Center-of-Sums defuzzification, Center-of-Largest-Area defuzzification, Height defuzzification, First-of-Maxima defuzzification, and Middle-of-Maxima defuzzification

Before we start to give an overview of these defuzzification methods, we will introduce some symbols that are needed to calculate a crisp defuzzification value. A linguistic variable is defined by $\langle X, \mathcal{L}X, \mathcal{X}, M_X\rangle$. Here X denotes the symbolic name of a linguistic variable, $\mathcal{L}X$ is the set of *linguistic values* that X can take on. We denote an arbitrary element of $\mathcal{L}X$ by LX. $\mathcal{X}$ is the physical domain over which X takes its quantitative (crisp) values. M_X is a semantic function which gives a 'meaning' (interpretation) of a linguistic value in terms of the elements of $\mathcal{X}$, i.e. $M_X : LX \to \widetilde{LX}$, where $\widetilde{LX}$ is a denotation for a fuzzy set defined over $\mathcal{X}$. Instead of $\widetilde{LX}$ we will also use μ_{LX}, i.e., the membership function without an argument. We can define a set of m rules as

if e_1 is $LE^{(k)}$ *and* ... *and* e_n is $LE_n^{(k)}$ *then* u is $LU^{(k)}$, $k = 1, \ldots, m$.

The firing of these rules with physical, crisp input values $e_1^*, \ldots, e_n^*$ will usually result in m clipped fuzzy sets denoted by $\widetilde{CLU}^{(1)}, \ldots \widetilde{CLU}^{(m)}$. We will use CLU as the general term. The *union* of these fuzzy sets will be denoted by $\tilde{U}$ or μ_U, $\tilde{U} = \bigcup_{k=1}^m \widetilde{CLU}^{(k)}$; the actual crisp defuzzification value we will denote by u^*. The *area* of the fuzzy set $\tilde{U}$ is defined as $\text{area}(\tilde{U}) = \int_{\mathcal{U}} \mu_U(u)\,\mathrm{d}u$, where $\int$ is the mathematical integral. The *height* of $\widetilde{CLU}^{(k)}$ is equal to the degree of match of the k-th rule antecedent, and will be denoted by f_k. The *peak value* of $\widetilde{CLU}^{(k)}$ is equal to the peak value of its unclipped version $\widetilde{LU}^{(k)}$ or $\mu_{LU}^{(k)}$. If $\widetilde{LU}^{(k)}$ is a triangular membership function, then its peak value is that domain element on $\mathcal{U}$ which has degree of membership 1.

Center-of-Area/Gravity This method (in the literature also referred to as Center-of-Gravity method) is the best well known defuzzification method. This method determines the center of the area below the combined membership function $\tilde{U}$. In the continuous case (the domain $\mathcal{U}$ is an interval) the defuzzified value u^* of $\tilde{U}$ is given by

$$u^*_{\text{CoA}} = \frac{\int_{\mathcal{U}} u \cdot \mu_U(u)\,\mathrm{d}u}{\int_{\mathcal{U}} \mu_U(u)\,\mathrm{d}u} = \frac{\int_{\mathcal{U}} u \cdot \max_k \mu_{CLU^{(k)}}(u)\,\mathrm{d}u}{\int_{\mathcal{U}} \max_k \mu_{CLU^{(k)}}(u)\,\mathrm{d}u}, \quad (1)$$

where $\int$ is the mathematical integral. In the discrete case ($\mathcal{U} = \{u_1, \ldots, u_\ell\}$), the integral $\int_{\mathcal{U}} \mu_U(u)\,\mathrm{d}u$ is replaced by the sum $\sum_{i=1}^{\ell} \mu_U(u_i)$, etc.

Center-of-Sums The motivation for using this method is to avoid the computation of $\tilde{U}$. The idea is to consider the contribution of the area of each $\widetilde{CLU}^{(k)}$ individually. Overlapping areas, if such exist, are reflected more than once by this method. The Center-of-Sums defuzzification method, for the discrete and continuous case respectively, is formally given by

$$u^* = \frac{\sum_{i=1}^{\ell} u_i \cdot \sum_{k=1}^{n} \mu_{CLU^{(k)}}(u_i)}{\sum_{i=1}^{\ell} \sum_{k=1}^{n} \mu_{CLU^{(k)}}(u_i)} \tag{2}$$

or

$$u^* = \frac{\int_{\mathcal{U}} u \cdot \sum_{k=1}^{n} \mu_{CLU^{(k)}}(u)\, du}{\int_{\mathcal{U}} \sum_{k=1}^{n} \mu_{CLU^{(k)}}(u)\, du}, \tag{3}$$

for the discrete and continuous case respectively.

Height Height defuzzification takes the peak value of each $\widetilde{CLU}^{(k)}$ and builds the weighted (w.r.t. f_k) sum of these peak values. Thus neither the support or shape of $\widetilde{CLU}^{(k)}$ play a role in the computation of u^*. The Height method is a very simple and quick method. Let $c^{(k)}$ be the peak value of $\widetilde{LU}$. Then the Height defuzzification method in a system of m rules gives $u^*_{\mathrm{HM}} = (\sum_{k=1}^{m} c^{(k)} \cdot f_k)/\sum_{k=1}^{n} f_k$.

Center-of-Largest-Area When the overall output fuzzy set $\tilde{U}$ is non-convex, i.e., consists of at least two convex fuzzy subsets, then the Center-of-Largest-Area method determines the convex fuzzy subset with the largest area and defines the crisp output value u^*_{CoLA} to be the Center-of-Area of this particular fuzzy subset. It is difficult to represent this defuzzification method formally, because this involves first finding the convex fuzzy subsets, then computing their areas, etc.

First-of-Maxima First-of-Maxima uses $\tilde{U}$ and takes the smallest value of the domain $\mathcal{U}$ with maximal membership degree in $\tilde{U}$. This is realized formally in three steps. Let $\mathrm{hgt}(U) = \sup_{u\in\mathcal{U}} \mu_U(u)$ be the highest membership degree of $\tilde{U}$, and let $D(\mathcal{U}) = \{u \in \mathcal{U} \mid \mu_U(u) = \mathrm{hgt}(U)\}$ be the set of domain elements with degree of membership equal to $\mathrm{hgt}(U)$. Then u^* is given by $u^*_{\mathrm{FoM}} = \inf_{u\in\mathcal{U}} D(\mathcal{U})$. The alternative of this method is called the Last-of-Maxima, where infimum is substituted extremum.

Middle-of-Maxima Middle-of-Maxima is very similar to First-of-Maxima or Last-of-Maxima. Instead of determining u^*_{MoM} to be the first or last from all values where $\mathcal{U}$ has maximal membership degree, this method takes the average of these values, i.e., $u^*_{\mathrm{MoM}} = (u^*_{\mathrm{FOM}} + u^*_{\mathrm{LOM}})/2$.

2 Comparison and Evaluation of Defuzzification Methods

In this section we will discuss the advantages and disadvantages of defuzzification methods. But before we can do this, we have to develop criteria an ideal defuzzification method should satisfy. Now it must be stated in advance that none of our defuzzification methods satisfies all criteria listed below, i.e., one has to weight these criteria for the application under concern in order to be able to make the right choice of the defuzzification method.

Some criteria for defuzzification methods are

1. Continuity A small change in the input of the fuzzy controller should not result in a large change in the output. For example, in the case of a two-input, one-output fuzzy controller, when two inputs $(e_1^*, \dot{e}_1^*)$ and $(e_2^*, \dot{e}_2^*)$ differ slightly, then the corresponding output values u_1^* and u_2^* should differ slightly too.

2. Disambiguity A defuzzification method is disambiguous if the algorithm to find u^* is well-defined. This criterion is not satisfied by Center-of-Largest-Area if there are two equally large areas.

3. Plausibility Every defuzzified control output has a horizontal component $u^* \in \mathcal{U}$, and a vertical component $\mu_U(u^*) \in [0,1]$. We define u^* to be plausible if it lies approximately in the middle of the support of $\tilde{U}$ and has a high degree of membership in $\tilde{U}$.

4. Computational complexity This criterion is particularly important in practical applications of fuzzy controller. The Height method, together with the Middle- and First-of-Maxima are fast methods, whereas the Center-of-Area method is slower. The computational complexity of Center-of-Sums depends on the shape of the output membership functions and whether max-min composition based inference or scaled inference is chosen.

A fifth criterion is the so-called weighting of the output fuzzy sets which constitutes the difference between the Center-of-Area and Center-of-Sums method. This criterion is handled separately from the former four criteria, because it is hard to say whether it is to be preferred or not. We consider it as a positive property.

5. Weight counting A defuzzification method is weight counting if it sums up the overlapping parts in the overall ouput fuzzy set $\tilde{U}$. Center-of-Sums and Height defuzzification are weight counting. Center-of-Area, for example, uses solely $\tilde{U}$ and therefore is not weight counting.

Table I gives an overview of these defuzzification methods and their performance with respect to these five criteria.

3 Defuzzification in Unbalanced Output Domains

Chen and Hsu [1] discuss an interesting problem in fuzzy logic with asymmetric output domains, i.e., output domains with fuzzy sets of different shape and support. Now let us consider such a domain in an extreme case, e.g., there are

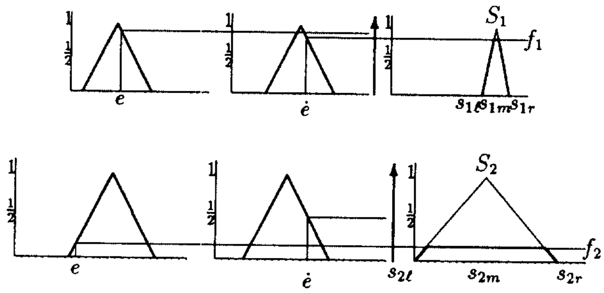

Figure 1. Two firing rules with different output fuzzy sets.

two rules firing like in Fig. 1. These rules can be expressed by

$$\textit{if}\ \langle \text{antecedent}_i \rangle\ \textit{then}\ u \text{ is } S_i, \qquad i = 1, 2.$$

where $\mu_{S_i}(x) = \Lambda(x; s_{i\ell}, s_{im}, s_{ir})$, i.e. S_i has a triangular membership function where $s_{i\ell}$ and s_{ir} have membership degree 0 and s_{im} has membership degree 1. The consequent of the first rule is a fuzzy set with a rather small support ($d_1 = s_{1r} - s_{1\ell}$ is small), whereas the second rule has a consequent with a very large support ($d_2 = s_{2r} - s_{2\ell}$ is big). This means that the first rule states that the output value is most certainly equal to s_{1m} and can have small deviations. The location of the output value of the second rule can not be given exactly, it can change rather much. This means that the first rule is a very good one and the second is a very bad one. However, when Center-of-Area, Center-of-Sums or Center-of-Largest-Area defuzzification is used, the crisp output value is overshadowed by the vague output value and will even not lie in the support interval of the first output fuzzy set, which is an unacceptable result. It is clear that in a situation like this the height method does not perform well too, because the it assumes that the support is the same for all the output fuzzy sets. The method proposed by Chen and Hsu [1] to solve this problem is some kind of normalizing the output fuzzy sets. They not normalize in advance, because in that case all the intersecting areas would change, but they do it afterwards. The main disadvantage of this method is that it ignores the fact that rules with 'crisper' outputs are *more important* than those with 'fuzzier' outputs.

Suppose now that there is a rule base with n rules

$$\textit{if}\ \langle \text{antecedent}_i \rangle\ \textit{then}\ u \text{ is } S_i, \qquad i = 1, \ldots, n,$$

where $\mu_{S_i}(x) = \Lambda(x; s_{i\ell}, s_{im}, s_{ir})$ like above. Let furthermore d_i be the length of the interval $[s_{i\ell}, s_{ir}]$, $d_i = s_{ir} - s_{i\ell}, i = 1, \ldots, n$. These values can be calculated in advance. Furthermore, let f_i ($i = 1, \ldots, n$) be the value with which rule R_i fires, e.g. $f_1 = 0.83$ and $f_2 = 0.17$ in Fig. 1. So $\tilde{S}_i$ is clipped at f_i. We call f_i the clipping-value. Then we can use the quotient $w_i = f_i/d_i, i = 1, \ldots, n$, as the weight that has to be attached to each clipped output fuzzy set, i.e., w_i is proportional to f_i and inversely proportional to d_i: w_i is maximal if the support of the output set is small and the clipping value is high, like the first rule in Fig. 1; w_i is minimal if the support of the output set is large and the clipping value is low, like the second rule in Fig. 1. So w_i can be considered as a quality measure of the clipped output fuzzy set. If w_i is high, then the information of the clipped output set is high, if w_i is low, then the information of the clipped output set is low.

The crisp defuzzified value u^* of this set of rules for particular input values is then given by

$$u^* = \left(\sum_{i=1}^{n} s_{im} \cdot w_i\right) \Big/ \sum_{i=1}^{n} w_i. \tag{4}$$

We will call this defuzzification method the ***Quality method***.

As an example, suppose there are two rules R_1 and R_2 with conclusions $\mu_{S_1}(u) = \Lambda(u; 7, 8.5, 10)$ and $\mu_{S_2}(u) = \Lambda(u; 2, 5.5, 9)$; and suppose they fire with values $f_1 = 0.8$ and $f_2 = 0.2$. This situation can be compared with Fig. 1. Then $w_1 = 0.8/3 \approx 0.267$ and $w_2 = 0.2/7 \approx 0.029$, i.e. the result of rule 1 is weighted approximately ten times more important as the result of rule 2. The defuzzified output value u^* is

$$u^* = \frac{s_{1m} \cdot w_1 + s_{2m} \cdot w_2}{w_1 + w_2} = \frac{5.5 \cdot \frac{0.2}{7} + 8.5 \cdot \frac{0.8}{3}}{\frac{0.2}{7} + \frac{0.8}{3}} \approx 8.2. \tag{5}$$

This means that a value very close to the peak value 8.5 of S_1 results.

An alternative approach is to additionally weight the supports of the output fuzzy sets with the following formula

$$u^* = \left(\sum_{i=1}^{n} \frac{s_{im} \cdot f_i}{d_i^{\xi}}\right) \Big/ \left(\sum_{i=1}^{n} \frac{f_i}{d_i^{\xi}}\right), \qquad \xi \geq 0. \tag{6}$$

For $\xi = 1$ this results in Eq. (4); for $\xi = 0$ this is the Height defuzzification method, the supports of the output fuzzy sets are not taken into account; for $0 < \xi < 1$ one gets an output value that lies between the Height method and the Quality method; and for $\xi > 1$ one obtains output values that are even more influenced by the rule with the highest weight w_i. We will call this defuzzification method the ***ξ-Quality method***.

Another alternative approach is to use a convex sum of the Center-of-Area method and the Quality method. Let u_1^* be the defuzzified output value obtained by the Center-of-Area method and let u_2^* be the defuzzified output value obtained by the Quality method, then the convex sum of these two is given by

$$u^* = \gamma \cdot u_1^* + (1-\gamma)\cdot u_2^*, \qquad 0 \le \gamma \le 1. \tag{7}$$

For $\gamma = 1$ this results in the Center-of-Area method, for $\gamma = 0$ this results in the Quality method. For values of γ between 0 and 1, u^* lies between these two methods.

Until now we have only considered membership functions with straight lines, but what happens when there are curved lines too? If we rewrite Eq. (4) into

$$u^* = \frac{\sum_{i=1}^{n} s_{im} \cdot w_i}{\sum_{i=1}^{n} w_i} = \frac{\sum_{i=1}^{n} \frac{s_{im} \cdot f_i^2}{\text{area}(S_i)}}{\sum_{i=1}^{n} \frac{f_i^2}{\text{area}(S_i)}} \tag{8}$$

where area(S_i) is the area under the original symmetric output fuzzy set $\tilde{S}_i$, i.e., let μ_{S_i} be the membership function of $\tilde{S}_i$, then

$$\text{area}(S_i) = \int_{\mathcal{X}} \mu_{S_i}(u)\mathrm{d}u, \tag{9}$$

which is equal to $\frac{1}{2} f_i d_i$ in the triangular case.

Table I shows the advantages of the ξ-Quality method in comparison to the six defuzzification methods given before. 'Quality regarding' is added as a criterion whether the defuzzification method takes into account unbalanced domains.

4 Asymmetric Membership Functions

We have considered output domains with symmetric output fuzzy sets and supports of different magnitude, but until now we have assumed that the membership functions were symmetric. However, when asymmetric membership functions like the one in Fig. 2 are used as well, the Heigth and Quality defuzzification methods are not performing well with respect to plausibility, neither do the First- and Middle-of-Maxima methods. These kinds of membership functions are better handled with the Center-of-Area method, because they do not focus on the point of membership degree 1, but on the area. Let $\widetilde{LU}^{(k)}$ be an asymmetric output fuzzy set, then we will call $v(LU^{(k)})$ the domain value of the center of area of $\widetilde{LU}^{(k)}$, i.e.,

	CoA	CoS	MoM	FoM	HM	CLA	QM
Continuity	++	++	--	--	--	0	++
Disambiguity	++	++	0	++	++	--	++
Plausibility	0	+	0	0	+	++	+
Comp. complexity	--	0	+	+	++	--	++
Weight counting	--	++	--	--	0	--	++
Quality regarding	--	--	0	0	-	--	++

Table I. A comparison of different defuzzification methods.

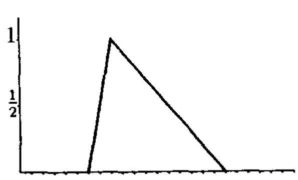

Figure 2. An example of an asymetric membership function.

$$v(LU^{(k)}) = \frac{\int_{\mathcal{U}} u \cdot \mu_{LU^{(k)}}(u)\mathrm{d}u}{\int_{\mathcal{U}} \mu_{LU^{(k)}}(u)\mathrm{d}u}, \tag{10}$$

and

$$v(LU^{(k)}) = \frac{\sum_{i=1}^{m} u_i \cdot \mu_{LU^{(k)}}(u_i)}{\sum_{i=1}^{m} \mu_{LU^{(k)}}(u_i)} \tag{11}$$

in the continuous and the discrete case ($\mathcal{U} = \{u_1, \ldots, u_m\}$) respectively.

We can use this value $v(LU^{(k)})$ in the Height and ξ-Quality method instead of the peak value. We then get the following formulas

$$u^* = \frac{\sum_{k=1}^{n} v(LU^{(k)}) \cdot \mu_{CLU^{(k)}}(u)}{\sum_{k=1}^{n} \mu_{CLU^{(k)}}(u)} \tag{12}$$

and

$$u^* = \frac{\sum_{k=1}^{n} v(LU^{(k)}) \cdot w_k}{\sum_{k=1}^{n} w_k}. \tag{13}$$

for the Height and ξ-Quality defuzzification methods respectively. Hence, the peak value $c^{(k)}$ is replaced by the center of area $v(LU^{(k)})$ of each individual fuzzy set. So when one wants to use one of these two formulas, one first has to calculate the center of area of each output fuzzy set. This is, however, no drawback for run-time behaviour.

References

[1] Chen, Y.-Y. & Hsu, S.-T. I., *Rules Aggregations in Fuzzy Ruled-Based Systems*, Internal Report National Taiwan University, 1991.

[2] Hellendoorn, H., *Reasoning with Fuzzy Logic*, Ph.D.-thesis Delft University of Technology, Delft, 1990.

[3] Hellendoorn, H., "Fuzzy Logic and Fuzzy Control." In: René van der Vleuten et al. (eds.), *Clear Applications of Fuzzy Logic*, (Proceedings IEEE-Symposium Delft, 17 Oct. 1991) Delft, pp. 57–82.

[4] Zadeh, L.A., "Fuzzy Sets," *Information and Control*, 8(1965)338–353.